The city of Alexandria (on the northern coast of Egypt) was founded by Alexander the Great in 332 B.C. Ptolemy I made Alexandria his capital and opened a university there, about 300 B.C. This university, called the 'Museum', soon had a library with more than 600,000 papyrus rolls. This library was destroyed by the Arabs in 641 A.D.

The first chair of mathematics at the Museum was occupied by Euclid. He wrote books on optics, music, and astronomy, but his fame rests on the *Elements,* a collection of 13 small books that present the 'elements' or introductory parts of the mathematics studied in Alexandria.

None of the theorems in the 13 books can be ascribed to Euclid himself. The Pythagoreans, including Archytas, were responsible for the contents of Books I, II, VI, VII, VIII, IX, and XI. Hippocrates was the genius behind Books III and IV. For Books V and XII we can thank Eudoxus. Books X and XIII are based on the work of Theaetetus.

Euclid's contribution was the logical organisation of the *Elements* — its axiomatic structure in which everything is carefully deduced from a small number of definitions and assumptions. This structure served as a model for Aquinas's *Summa Contra Gentiles,* for Newton's *Principia,* and for Spinoza's *Ethics.* The *Elements* has been the most influential textbook in history.

Our contemporary axiomatic approach is to take sets or natural numbers as basic, but Euclid, perhaps because he did not know how to give a set theoretical or arithmetic treatment of irrationals, started with points and lines. Euclid expressed the laws of arithmetic geometrically. For Euclid, a number is a line segment.

Another difference between us and Euclid is that Euclid regarded his starting assumptions not as mere hypotheses, but as truths. He intended to instantiate the ideal described by Aristotle at the beginning of the *Posterior Analytics:* sure knowledge is obtained by the rigorous deduction of the consequences of basic truths. These truths are either definitions or existence assertions.

Euclid begins the *Elements* with a list of 23 definitions. The first is that 'a point is that which has no part'. These are followed by 5 postulates governing what can be constructed and hence what has mathematical 'existence'. For example, the first postulate says: 'to draw a straight line from any point to any point'. In other words, given any two points, there is a straight line that passes through both of them.

Euclid's fifth postulate is the famous Parallel Postulate:

> if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

In other words, if A and D are points on the same side of line BC, and $\angle ABC + \angle DCB < 180^\circ$ then there is a point on that same side of BC where BA (suitably lengthened if necessary) meets CD (suitably lengthened if necessary).

For a long time mathematicians tried to prove the fifth postulate from Euclid's other starting assumptions (including some tacit ones), but in the nineteenth century, Eugenio Beltrami (1835–1900) and others showed that this cannot be done. The geometry obtained by adding the negation of the fifth postulate to Euclid's other axioms is a weird, but consistent, geometry called *hyperbolic geometry.* In hyperbolic geometry there are no squares, and not every triangle has a circumcircle.

Following Euclid's 5 postulates are his 5 common notions, or logical truths. The first is 'things which are equal to the same thing are also equal to one another.' The fifth common notion is 'the whole is greater than the part'. Note that in modern set theory there is a sense in which the whole set of natural numbers is not greater than its part the set of evens, since the two sets can be placed in one-to-one correspondence.

Following the 5 common notions are the 48 propositions of Book I, culminating in the theorem of Pythagoras and its converse. Before deducing its properties, Euclid is careful to show that it is indeed possible to construct a square on the hypotenuse.

Excerpts from Anglin, W.S. (1994). *Mathematics: A Concise History*. New York, NY. Springer-Verlag.

Book II gives a geometric treatment of some basic algebraic identities, such as the distributive law. It also includes the Law of Cosines.

Book III derives the basic properties of the circle. Euclid attempts to give rigorous proofs of his assertions, not merely relying on the diagrams. For example, he offers a proof of the fact that a point on a chord lies in the interior of the circle.

Book IV gives constructions for the regular pentagon and the regular 15-gon. This achievement was not surpassed until 1796, when Carl Friedrich Gauss (1777–1855) found a construction for the regular 17-gon.

In Book V, Euclid uses Eudoxus's definition of proportion, together with the 'axiom of Archimedes', to deduce a basic field arithmetic for line segments. The commutativity of multiplication is proved in Proposition V 16.

In Book VI Euclid studies similar (equiangular) triangles and proves that the length of a circular arc is proportionate to the angle it subtends at the centre of the circle.

Books VII to IX are on number theory. Included are proofs for Euclid's algorithm (VII 2), the unique factorisation of square-free integers (IX 14), the infinitude of primes (IX 20), and the formula for even perfect numbers (IX 36).

Book X investigates expressions like

$$\sqrt{7+2\sqrt{6}}$$

reducing them, if possible, to expressions with fewer square root signs (e.g., $1+\sqrt{6}$).

Book XI presents the basic theorems of solid geometry. Euclid constructs a cone by rotating a right triangle about one of its sides. This takes him beyond straightedge and compass constructions. Indeed, by intersecting his cones with planes, he could have constructed the parabolas that Menaechmus used to duplicate the cube. Straightedge and compass constructions are only a proper subset of the constructions found in Euclid.

Book XII is the masterpiece of Eudoxus. Without the aid of calculus, he manages to give a rigorous treatment of the volumes of the pyramid, cone, and sphere.

Book XIII is the apex of the *Elements.* For each of the 5 regular polyhedra, Euclid derives the ratio of its side to the radius of the sphere in which it can be inscribed. Euclid also proves that there are no other regular polyhedra than the 5 known to the Pythagoreans.

The school established by Euclid in Alexandria produced some first-rate mathematicians: Aristarchus, Archimedes, Apollonius, and Eratosthenes.

Aristarchus (310–250 B.C.) came from Samos, as had Pythagoras. He gave an important application of mathematics to astronomy. Let SEM be the triangle whose vertices are the sun, earth, and moon (respectively). Aristarchus reasoned that when the moon is at its first quarter, $\angle SME = 90°$. That is why we see exactly half the part of the moon's surface that faces the earth. When the moon is at its first quarter, one can see the sun and moon together in the sky, at the same time. Thus Aristarchus was able to measure $\angle SEM$. Using a drawing of a right-angled triangle with that same angle, Aristarchus found the ratio ES/EM. Without a telescope or space ship, he discovered that the sun is ES/EM times further from the earth than the moon is. (Aristarchus thought that $ES/EM = 20$, because he failed to measure $\angle SEM$ with sufficient accuracy. Actually, $ES/EM = 300$.)

Thanks to observations of solar eclipses, Aristarchus knew that the apparent diameter of the moon is equal to that of the sun. From this he deduced that the ratio of the sun's diameter to the moon's diameter is also ES/EM.

By observing the shadow of the earth on the moon during lunar eclipses, one can calculate the relative sizes of earth and moon. Aristarchus used this technique and his previous results, to calculate the ratio of the sun's diameter to the earth's diameter. As we shall see, Eratosthenes had a way of determining the size of the earth. The ancient Greeks were thus able to arrive at a knowledge of the size of the sun.

Aristarchus based his work on the following correct assumptions about the solar system:

(1) The sun, moon, and earth are spheres;
(2) The earth goes around the sun, and the moon around the earth;
(3) Light travels in straight lines;
(4) The moon's light is a reflection of the sun's light;
(5) Solar eclipses are caused by the moon's blocking the sun's rays to the earth, and lunar eclipses are caused by the earth's blocking the sun's rays to the moon.

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