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Two lengths a and b are *commensurable* if there are positive integers p and q such that $a/b = p/q$. When the Pythagoreans claimed that all things are numbers, they meant to imply that all pairs of lengths are commensurable. For the Pythagoreans, 'number' meant 'rational number'.

Unfortunately, they soon discovered that the diagonal of a unit square is not commensurable with its side. A proof of this is found in Aristotle's *Prior Analytics* 41a23-30. Let $ABCD$ be a square whose sides have length 1. By the theorem of Pythagoras, the diagonal AC measures $\sqrt{2}$. Suppose

$$\sqrt{2} = AC/AB = p/q$$

where p and q are positive integers. We may assume that p and q are relatively prime (have no common factor). In particular, we may assume that they are not both even.

Now $p^2 = 2q^2$, so that p^2 is even. As the Pythagoreans well knew, the square of an odd number is odd, while the square of an even number is even. Thus, from the fact that p^2 is even, it follows that p is even. Suppose $p = 2r$. Then $(2r)^2 = 2q^2$ and hence $q^2 = 2r^2$. But this means that q is even as well. Contradiction. The assumption that AC and AB are commensurable leads to an absurdity.

The Pythagoreans tried, at first, to keep this discovery a secret, as it undermined their philosophy. Some say it was Hippasus (470 B.C.) who leaked the secret and that he drowned as a punishment for having done so.

The Greeks did not know how to handle $\sqrt{2}$ in an arithmetic or algebraic fashion. They did, however, know that it was a length (of a diagonal), and they turned to geometry for an understanding of it. The problem of incommensurables was one reason why they preferred to do algebra in a geometric manner. For example, the ancient Greeks thought of the distributive law $a(b+c) = ab + ac$ as an addition rule for areas of rectangles with the same width a.

Excerpts from Anglin, W.S. (1994). *Mathematics: A Concise History*. New York, NY. Springer-Verlag.

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Motion is impossible, said Zeno, because a moving object must first go half the total distance it will travel, then half the remaining distance, and so on, forever. If a point moves from position 0 to position 1 on the number line, it first reaches position 1/2, then position 3/4, then position 7/8, and so on. At the nth stage, it is at position $1 - \frac{1}{2^n}$. From the fact that there is no n such that $1 - \frac{1}{2^n} = 1$, it follows that that moving point never reaches position 1. It just cannot get through the infinite number of stages necessary to do so. Hence there is no motion, motion from 0 to 1 being typical of any motion whatsoever.

In modern physics, we counter this argument by asserting that, indeed, the point can and does traverse each of the infinite number of intervals from $1 - \frac{1}{2^n}$ to $1 - \frac{1}{2^{n+1}}$ for $n = 1, 2, 3, \ldots$ — *ad infinitum*. There is no n such that the moving point does not cross position $1 - \frac{1}{2^n}$. Starting from the premiss that there is motion, modern physicists invoke the infinite to explain it. Like Zeno, they assume that motion is continuous, but, unlike Zeno, they are willing to say that a moving object does pass over an infinite number of points. Zeno rejected the infinite, and so he rejected motion too. Modern physicists accept motion, and so they accept the infinite too.

The famous runner Achilles and his rival (usually thought to be a tortoise) are racing along the positive number line. Achilles starts at position 0, but the tortoise has a head start, beginning at position 1. Since Achilles runs twice as fast as the tortoise, one might expect him to overtake the tortoise at position 2. However, when Achilles arrives at position 1, the tortoise is already at position $1 + \frac{1}{2}$; when Achilles reaches position $1 + \frac{1}{2}$, the tortoise has raced on to position $1 + \frac{1}{2} + \frac{1}{4}$; and so on. When Achilles finally gets to position $2 - \frac{1}{2^n}$, for large n, the tortoise is still ahead, at position $2 - \frac{1}{2^{n+1}}$. Despite the appearances, which lead us to believe there is motion, Achilles will never catch up to the tortoise.

In this second argument, Zeno again assumed, as we do, that space and time are continuous, and that, if there is motion, there is uniform motion. Zeno also assumed, unlike us, that Achilles and the tortoise can never 'get through' the infinite number of stages into which Zeno analysed their motion.

For modern physics, precisely, motion typically consists of the occupation of infinitely many distinct locations at infinitely many distinct instants — all within a finite time interval. Because we accept the infinite, we do not find Zeno's argument troubling. However, if someone rejected the infinite, he or she would, indeed, have to reject the possibility of continuous motion.

Antiphon (425 B.C.), another sophist, asserted the equality of all human beings.

In mathematics, he was the first person to suggest that the area of a circle be calculated in terms of regular polygons inscribed in it.

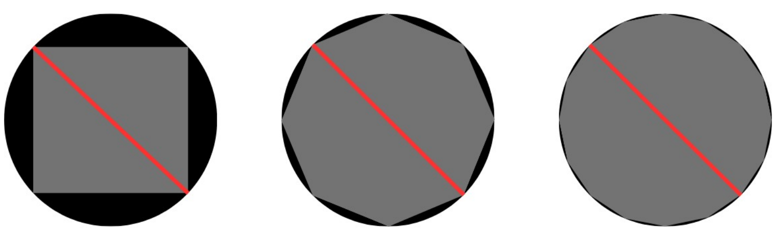
An inscribed square takes up more than 1/2 the area of a circle, while an inscribed regular octagon takes up more than 3/4 the area of a circle. As the ancient Greeks realised (see the *Elements* XII 2), one can use what we now call mathematical induction to show that an inscribed regular 2^n-gon takes up more than $1 - \frac{1}{2^{n-1}}$ of the area of a circle.

If we inscribe a regular 2^n-gon in a circle, its longest diagonals are diameters of that circle. The ancient Greeks knew that the area of a regular 2^n-gon is proportionate to the square on its longest diagonal, and from this it follows that, *in so far as a circle is like a regular* 2^n*-gon*, its area is proportionate to the square on its diameter:

$$\text{area of circle } = k(2r)^2 = (4k)r^2$$

where r is the radius of the circle.

Antiphon argued for this (correct) conclusion by boldly asserting that a circle simply *is* a regular polygon with an infinite number of sides.



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The ancient Greeks searched for a way of using a straightedge and a compass to trisect an arbitrary angle and draw a segment of length $\sqrt[3]{2}$. They also tried to 'square the circle', that is, construct a segment of length $\sqrt{\pi}$. Finally, they struggled to find straightedge and compass constructions for regular polygons with 7, 9, 11, 13, and 17 sides. In all this they failed, but it was not proved until the nineteenth century that the reason for their failure was that all these problems are impossible — except one. In 1796 Gauss discovered a straightedge and compass construction for the regular 17–sided polygon. It was this discovery, the first advance on Greek construction problems in 2000 years, that motivated Gauss to devote himself to mathematics.

Sadly, it is now possible to obtain a Ph.D. in mathematics and not know that Euclid lived in Alexandria, Egypt, about 300 B.C., and wrote a book called the *Elements*. When we do geometry today, we usually start with a plane that already contains a point corresponding to every ordered pair of reals. Euclid was more parsimonious. He started with just two points (corresponding to (0, 0) and (1, 0)) and then constructed, one by one, just enough extra points to meet his immediate needs.

The rules for construction were strict.

(1) If A and B are previously given or constructed points, you can 'join AB', constructing the line segment AB; if this segment intersects any previously constructed line segments or circles, you have thereby constructed the points of intersection.

(2) If AB is a previously constructed segment and O is a previously given or constructed point, you can draw a circle (that is, a circumference) with centre O and radius AB; if this circle intersects any previously constructed line segments or circles, you have thereby constructed the points of intersection.

(3) If AB is a previously constructed segment, you can lengthen, or 'produce', it in either direction to meet a previously constructed segment or circle (assuming that that segment or circle lies 'in its way'), and thereby construct a point.

(4) The only way to construct anything is to apply the above rules a finite number of times.