A *polyhedron* is a solid whose surface consists of polygon faces. A polyhedron is *regular* or *Platonic* if its faces are congruent regular polygons and if its polyhedral angles are all congruent. Five regular polyhedra are the following.

The **cube** is bound by 6 squares, with 3 squares at a vertex.

The **tetrahedron** is bound by 4 equilateral triangles, with 3 triangles at a vertex.

The **octahedron** is bound by 8 equilateral triangles, with 4 triangles at a vertex.

The **icosahedron** is bound by 20 equilateral triangles, with 5 triangles at a vertex.

The **dodecahedron** is bound by 12 regular pentagons, with 3 pentagons at a vertex.

Pythagoras himself knew of the first four of these solids, but it was Hippasus (470 B.C.) who discovered the dodecahedron. On one account, Hippasus was expelled from the Pythagorean order for failing to attribute the discovery to the Master.

A proof that there are only these 5 regular polyhedra is found in Euclid's *Elements* (300 B.C.). This proof is based on the fact that if q regular p-gon faces meet at a vertex, then the sum of the q angles in the q faces is less than 360°. This is proved rigorously in Proposition 21 of Book IX of the *Elements,* but it can be seen intuitively by imagining someone cutting the q edges and flattening the polyhedral angle. For example, if one cuts the three edges at the vertex of a cube and flattens the solid angle, the three right angles still have a common vertex, and one can see that their sum is less than a complete revolution.

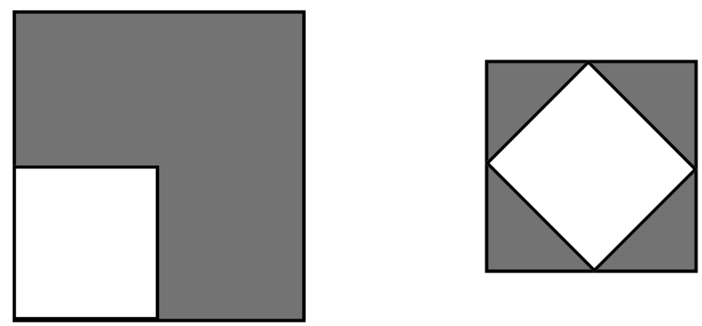
Since one can dissect a polygon with p sides into $p-2$ triangles, the sum of the angles of a polygon with p sides is $(p-2) \times 180°$. Each angle of a regular p-gon is thus $\frac{(p-2)\times 180°}{p}$. For a regular polyhedron whose faces are regular p-gons, with q p-gons meeting at a vertex, we thus have

$$q\frac{(p-2) \times 180°}{p} < 360°$$

or $\frac{1}{2} < \frac{1}{p} + \frac{1}{q}$. Of course, p and q are each at least 3, and, moreover, they cannot both be greater than 3 (lest $\frac{1}{2} > \frac{1}{p} + \frac{1}{q}$). So there are only 5 possibilities for p and q, one for each of the 5 regular solids already given.

Socrates (469–399 B.C.) was Plato's mentor. He was not a mathematician but, as Plato portrays him in the *Meno,* he made use of mathematics in philosophy. In the course of a conversation about virtue with Meno, Socrates has one of Meno's uneducated slave boys 'double the square'. The boy at first thinks that one doubles the area of a square by doubling its side, but Socrates soon leads him to see his mistake. Then Socrates shows him the figure of a square with the midpoints of its four sides joined to form a smaller square. Socrates then gets the boy to 'remember' that it is the square on the diagonal of the original square that has double its area (see the figure).

The mathematics was not very revolutionary, but the rest of Socrates's message was. Here was a boy quite capable of learning mathematics, and Meno was failing to educate him.



Excerpts from Anglin, W.S. (1994). *Mathematics: A Concise History*. New York, NY. Springer-Verlag.

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Plato (427–349 B.C.) was a student of Socrates. After Socrates's execution in 399 B.C., Plato travelled to North Africa, where he studied with Theodorus of Cyrene (Lybia), a mathematician who proved the irrationality of the square roots of 3, 5, 6, 7, 10, 11, 13, 14, 15, and 17.

At age 40, Plato visited Italy, and spent some time with the Pythagorean mathematician Archytas (428–347 B.C.). It was Archytas who first found a construction for the $\sqrt[3]{2}$. Plato's trip to Italy came to a sudden end when his enemy Dionysius I, ruler of Syracuse, sold him into slavery! Happily, one of Plato's friends ransomed him, and he returned to Athens.

About 380 B.C., Plato found the Academy. At the entrance of this research institute was the inscription:

LET NO ONE IGNORANT OF GEOMETRY ENTER HERE!

Plato was a realist: he held that reality exists independently of the human mind. He was also a correspondence theorist: he held that a statement is true just in case it correctly describes the actual state of affairs (in mind-independent reality). Not surprisingly, Plato attacked the relativism of Protagoras, according to which anything 'is to me such as it appears to me, and is to you such as it appears to you'.

Plato believed that the objects in the universe fall into two very different classes, the material and the immaterial. Objects such as the sun, that bed, and Diana's body belong to the class of material things. Objects such as goodness, that circle, and Diana's soul belong to the class of immaterial things. A drawing of a square belongs to the material realm, but the square itself belongs to the immaterial realm. Plato says of geometry students that they

> make use of the visible forms and talk about them, though they are not thinking of them but of those things of which they are a likeness, pursuing their inquiry for the sake of the square as such and the diagonal as such, and not for the sake of the image of it which they draw (*Republic* 510d).

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In the *Posterior Analytics,* Aristotle formulates what we call the deductive method. It was adopted by Euclid and has always been an essential characteristic of mathematics. This method consists of starting with propositions called axioms and then proving propositions called theorems. Each statement in a proof has to be justified either by an axiom or by a previously proved theorem or by a principle of logic.

For Aristotle, the axioms of mathematics are truths, and hence the theorems are also truths. Aristotle does not say, 'if there is a triangle with such and such a property, then the sum of its angles is two right angles'. Rather, he says, 'the triangle in virtue of its own nature contains two right angles' (*Metaphysics* IV 3).

Aristotle also did pioneering work in modal logic. In *De Interpretatione* 12 and 13, he notes the following implications.

(1) If it is possible that p is not the case, then it is not necessary that p be the case.

(2) If it is not possible that p be the case, then, necessarily, p is not the case.

For example, if it is not possible to pass the exam then, necessarily, you will fail it.

On Aristotle's scheme, every statement falls into exactly one of the following three categories.
(A) It is necessarily true.
For example: $2 + 5 = 7$.
(B) It is necessarily false (or impossible).
For example: This dog is actually a telephone number.
(C) It is contingent.
For example: Hitler invaded Russia. The French never fought the British.

Excerpts from Anglin, W.S. (1994). *Mathematics: A Concise History*. New York, NY. Springer-Verlag.