Proposition 27 of Book I of the *Elements* tells us that if the 'alternate angles' are equal then the lines are parallel: if $\angle AGH = \angle GHD$ then AB and CD are parallel (see the Figure on the next page). Proof: if AB (produced) does meet CD (produced) at X then GHX is a triangle in which the 'exterior angle' $\angle AGH$ is not greater than the 'interior and opposite angle' $\angle GHD$ — against Proposition 16.

In Proposition 29, Euclid uses the fifth postulate for the first time. This is to prove the converse of Proposition 27. Suppose AB is parallel to CD. To obtain a contradiction, suppose $\angle AGH \neq \angle GHD$, but, say, $\angle AGH > \angle GHD$. Then

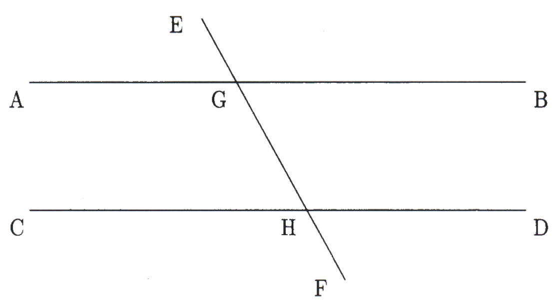
$$\angle GHD + \angle HGB < \angle AGH + \angle HGB = 180^\circ$$

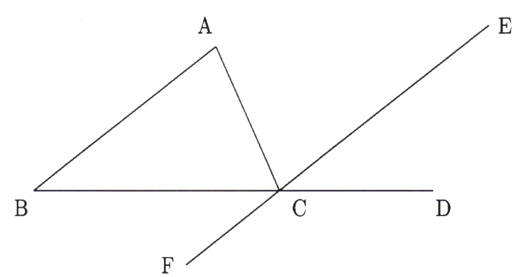
Thus, by the fifth postulate, AB meets CD. Contradiction. Thus $\angle AGH = \angle GHD$.

Proposition 32 shows that the sum of the angles in any triangle is two right angles. Indeed, if FCE is parallel to AB then $\angle ABC = \angle BCF = \angle ECD$ (Prop. 29, 15). Also $\angle BAC = \angle ACE$. Thus

$$\angle ABC + \angle BAC + \angle ACB = \angle ECD + \angle ACE + \angle ACB = 180^\circ$$

In non-Euclidean geometry, the sum of the angles of a triangle is not two right angles.





Eudoxus (405–355 B.C.) came from Cnidus, a small Greek island near present-day Turkey. He distinguished himself in astronomy, medicine, geography, philosophy, and, of course, mathematics.

As a young man, Eudoxus studied at Plato's Academy, commuting on foot from Piraeus, the harbour district. Later he engaged in a philosophical controversy with Plato. Eudoxus was a hedonist, but Plato put wisdom above pleasure. (See Plato's *Philebus* and Aristotle's *Nicomachaean Ethics* 1101b27 and 1172b9 for an account of this debate.)

Eudoxus was responsible for the material in Books V and XII of Euclid's *Elements*. Book V is a theory of proportion. We would define 'a is to b as c is to d' (written $a : b :: c : d$) to mean $a/b = c/d$. However, this definition presupposes our real number field. It presupposes that we already have some way of understanding what it is to multiply or divide arbitrarily given irrational numbers. Eudoxus, however, was starting from scratch. He could not use multiplication or division to define proportion because it was part of his program to define multiplication and division in terms of proportion. The definition of proportion on which he based his presentation of the number system was the following:

$$a : b :: c : d$$

iff, for any positive integers p and q, (1) $pa > qb$ iff $pc > qd$, and (2) $pa = qb$ iff $pc = qd$, and (3) $pa < qb$ iff $pc < qd$.

(Eudoxus assumed that all the numbers were positive.)

Modern		**Eudoxus**	
a : b :: c : d		a : b :: c : d	
2 : 3 :: 4 : 6		2 : 3 :: 4 : 6	
2/3 = 4/6	p2>q3 iff p4>q6	p2=q3 iff p4=q6	p2<q3 iff p4<q6
	p=5 q=2	p=3 q=2	p=2 q=5
	5*2>2*3 iff 5*4>2*6	3*2=2*3 iff 3*4=2*6	2*2<5*3 iff 2*4<5*6
	10>6 iff 20>12	6=6 iff 12=12	4<15 iff 8<30

Excerpts from Anglin, W.S. (1994). *Mathematics: A Concise History*. New York, NY. Springer-Verlag.

Eudoxus gave the following proof that the area of a circle is proportionate to its diameter squared. Suppose k is the area of the circle with diameter 1. (Hence k is the number we today call $\pi/4$.) Let c be a circle with diameter d. To obtain a contradiction, suppose

$$kd^2 < \text{ area } c$$

Let regular 2^n-gons be inscribed in both circles, where n is so large that

$$\frac{1}{2^{n-1}} \text{ area } c < \text{ area } c - kd^2$$

(This is possible according to the 'axiom of Archimedes', which, in fact, shows up in Aristotle and goes back at least to Eudoxus himself.) Then

$$\left(1 - \frac{1}{2^{n-1}}\right) \text{ area } c > kd^2$$

Now, as Antiphon realised, an inscribed regular 2^n-gon takes up more than $1 - 1/2^{n-1}$ of the area of a circle. Hence

$$\text{area of } 2^n\text{-gon in } c > kd^2$$

Antiphon also knew that the area of the 2^n-gon inscribed in c is d^2 times bigger than the one inscribed in the circle with unit diameter. Thus

$$kd^2 > \text{ area of } 2^n\text{-gon in unit diameter circle} \times d^2 > kd^2$$

Contradiction.

Similarly, we get a contradiction if we assume that $kd^2 >$ area c. From this it follows that $kd^2 =$ area c.

Excerpts from Anglin, W.S. (1994). *Mathematics: A Concise History*. New York, NY. Springer-Verlag.

Menaechmus discovered the conics, defining them as 'sections' of a cone and deriving equivalents of their analytic geometry formulas. For example, he defined a parabola as the intersection of a right circular cone and a plane parallel to a straight line in the (surface of) the cone.

Suppose the plane cuts the cone at points V, P, and Q, where V is the vertex of the parabola, and P and Q are points opposite each other on the cone. There is a circle $BPCQ$ that is in the (surface of) the cone, at right angles to the cone's axis, and is such that its diameter BC is perpendicular to PQ. If BC meets PQ at M, then VM is the axis of symmetry of the parabola.

Let A be the vertex of the cone, and let W be the point on the cone opposite V. Then triangle VMB is similar to triangle AWV (since VM and AW are parallel). Hence $VM/BM = AW/VW$, this being a constant independent of P. By the theorem of Thales, $\angle BPC$ is right, and thus $PM^2 = BM \times MC$. Since $VMCW$ is a parallelogram, we have $MC = VW$. Hence

$$\frac{VM}{PM^2} = \frac{VM}{BM \times MC} = \frac{VM}{BM \times VW} = \frac{AW}{VW^2}$$

which is a constant k, independent of the choice of P. This yields $VM = kPM^2$, which is essentially the same as the analytic geometry formula for the parabola.

Menaechmus used the conics to 'double the cube'. To do this, he may have used the fact we express as follows: the parabolas $y = \frac{1}{2}x^2$ and $x = y^2$ meet at a point whose y-coordinate is $\sqrt[3]{2}$.