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Euclid's first proposition is 'to construct an equilateral triangle'. Whereas today we think in terms of a plane that is an infinite set of points and already contains an infinite number of equilateral triangles, Euclid's plane is at first empty (except for a couple of 'starting points'). Later it contains points and lines and triangles — but only those that have been constructed, one by one, using straightedge and compass.

To construct an equilateral triangle on starting segment AB, Euclid constructs two circumferences, one with centre A and radius AB, and the other with centre B and radius BA. He assumes that they do not just pass through each other without touching but meet in a point C. He then joins C to A and to B. Of course, it is easy to show that ABC is equilateral: $AC = AB = BA = BC$.

In Proposition 4, Euclid shows that if ABC and DEF are triangles such that $\angle ABC = \angle DEF$, $AB = DE$, and $BC = EF$, then ABC is congruent to DEF. (That is, each side or angle of ABC has the same size as the corresponding side or angle of DEF, and their areas are equal too.) This is the side–angle–side or SAS theorem. Its proof is not rigorous — it involves motion — and modern geometers prefer to take the SAS theorem as another postulate. (Note that in saying ABC and DEF are congruent, we arrange the letters so that the angles at the corresponding vertices are equal: $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$.)

Proposition 8 shows that if $AB = DE$, $BC = EF$, and $CA = FD$ then triangles ABC and DEF are congruent. For if they are not, there is a triangle $A'EF$, with A' and D on the same side of EF, which is congruent to ABC but distinct from DEF. Since $A'E = AB = DE$, it follows that $\angle A'DE = \angle DA'E$. Similarly, $\angle A'DF = \angle DA'F$. But $\angle A'DE > \angle A'DF$ and hence $\angle DA'E > \angle DA'F$. Contradiction. Proposition 8 is the 'side-side-side' or SSS congruence theorem.

Proposition 47 is the theorem of Pythagoras. The diagram is sometimes called the 'Bride's Chair' (see the Figure on the next page). To prove that if triangle ABC has a right angle at A then $BC^2 = AB^2 + AC^2$, Euclid reasons as follows. By SAS, triangles FBC and ABD are congruent. Thus, using Proposition 41 twice,

$$AB^2 = \text{ area } AGFB = 2 \times \text{ area } FBC = 2 \times \text{ area } ABD = \text{ area } BDLJ$$

Excerpts from Anglin, W.S. (1994). *Mathematics: A Concise History*. New York, NY. Springer-Verlag.

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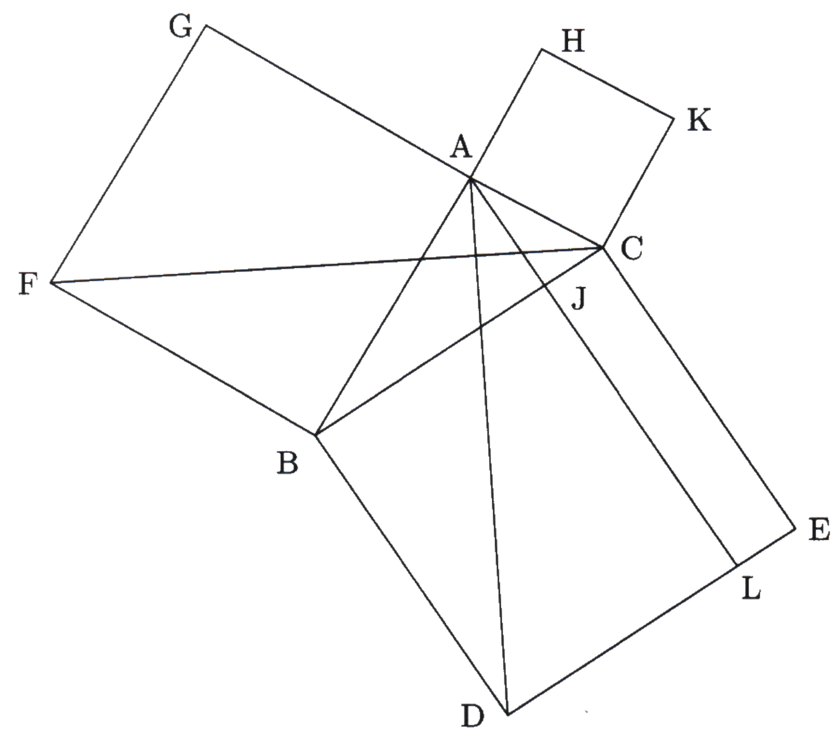
Apollonius (260–190 B.C.) came from Perga in the south of what is now Turkey. He wrote a book on conics that contained 400 theorems. Three of the things he is famous for are the following.

(1) Apollonius discovered that if the same sphere passes through the vertices of an icosahedron and the vertices of a dodecahedron, then

$$\frac{\text{surface area of dodecahedron}}{\text{surface area of icosahedron}} = \frac{\text{volume of dodecahedron}}{\text{volume of icosahedron}}$$

(2) Apollonius was the first to suggest that the moon and planets move in epicycloids. This was an incorrect but very influential theory.

Apollonius wrote a lost treatise on 'Tangencies' in which he gave a straightedge and compass construction for a circle tangent to three given circles. Using inversion, there is an easy way to do this, and this may have been the way actually used by Apollonius.



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Archimedes of Syracuse (287–212 B.C.) was the greatest mathematician and physicist before Isaac Newton. Many stories are told about Archimedes. One story relates that, while he was bathing, Archimedes suddenly discovered a simple way of determining the ratio of gold to silver in a gold-silver alloy. Elated by his insight, he leapt from the bath, and ran through the streets of Syracuse, shouting *Eureka!,* which means *I found it!* Archimedes, however, had forgotten to put on his clothes!

It was no accident that Archimedes made his discovery in the bath. Suppose you have an m kg crown made of gold and silver. Suppose you wish to determine the number x of kilograms of gold that the smith has put in it. If g is the density of gold and s is the density of silver, the volume of the crown is

$$v = \frac{x}{g} + \frac{m-x}{s}$$

What Archimedes realised was that, by immersing the crown in a rectangular bath tub, and observing the increase in the water level, you can determine its volume v. Then, solving the equation for x, you can obtain the mass of gold in the crown.

Thanks to his discovery, Archimedes was able to tell his friend, King Hieron, that the smith had cheated the king by charging him for pure gold, while in fact using a certain percentage of silver in the royal crown.

Archimedes wrote on many subjects, often solving problems by using what we would call calculus. He is thus, in a sense, one of the creators of that branch of mathematics. For example, he used calculus-like techniques to give the first proofs of many of our basic formulas, such as the formula for the area of a circle. As another example, he used calculus-like techniques to show that if two cylinders, each of radius 1, intersect each other at right angles, then their common volume is 16/3. (Note that a cross-section of the common volume, taken parallel to the plane containing the two axes of the cylinders, is a square. One 'adds up' these squares to get the volume. See the second edition of Schaum's *Calculus,* page 181.)

In geometry, Archimedes studied an area called the 'arbelos'. Let B be a point in straight line AC. Construct three semicircles with diameters AB, BC, and AC, all on the same side of AC. The area that is in the semicircle on AC but not in either of the two smaller semicircles is the *arbelos* (or 'shoemaker's knife'). Archimedes found the area of the arbelos:

> At B erect a perpendicular to AC, to meet the large semicircle on AC at W. Then the area of the arbelos equals the area of the circle with diameter BW.

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Eratosthenes of Cyrene (in North Africa) (275–195 B.C.) was chief librarian at Alexandria. He was interested in philosophy, poetry, history, philology, geography, astronomy, and mathematics.

Eratosthenes suggested a method for making a list of all prime numbers. This method, called the *Sieve of Eratosthenes,* works as follows. Start with the sequence of positive integers ≥ 2. Underline the 2, and cross out all the other multiples of 2. Go to the smallest positive integer n (on this list) which is neither underlined, nor crossed out, underline n, and cross out all the other multiples of n. Keep repeating the preceding step. In the end, the underlined numbers form a complete list of primes.

To measure the earth, Eratosthenes correctly assumed that since the sun is so far from the earth, those of its rays that hit the earth can be regarded as parallel. (Here he used the result of Aristarchus.) Eratosthenes knew that Syene (present-day Aswan) is on the Tropic of Cancer. That is, at noon on midsummer's day (June 21), the sun is directly overhead. At Alexandria, however, at noon on midsummer's day, the sun is 7° away from the point directly overhead. Since Alexandria is due north of Syene, the arc on the earth's surface between Alexandria and Syene subtends an angle of 7° at the earth's centre (see the Figure above). Eratosthenes knew the distance from Alexandria to Syene. In our metric units, it is 800 km. Using Euclid's theorem (VI 33) that the length of an arc is proportionate to the angle it subtends at the centre of the circle, Eratosthenes concluded that the circumference of the earth is, in effect,

$$\frac{360^\circ}{7^\circ} \times 800 = 41,000 \text{ km}$$

Hence its radius is 6500 km.

Using the work of Aristarchus, Eratosthenes was able to calculate the size of the moon and the sun. Comparing their apparent sizes to their actual sizes, he was able to discover the distance to the moon and the distance to the sun — all without using any of our technology.