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Hipparchus (180–125 B.C.) came from Nicaea, the town near present-day Istanbul, which was to be the site of the great pro-monotheistic council of 325 A.D. Hipparchus was an astronomer. He calculated the duration of the year to within 6 minutes.

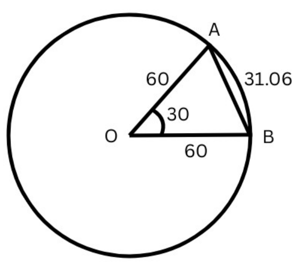
Hipparchus was the father of trigonometry. He drew up a table giving, for each whole number angle with vertex at the centre of a circle of radius 60, the length of the chord it cuts off that circle. For example, suppose $\angle AOB = 30°$, with O the centre of the circle and $OA = OB = 60$. Then the chord in question is the segment AB. This has length 31.06, so that in Hipparchus's table, we find

$$\text{chord}(30°) = 31.06$$

In modern terms, $\text{chord}(x) = 120 \times \sin(\frac{x}{2})$. To construct his table, Hipparchus used formulas we would express as

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$2\sin^2(\frac{x}{2}) = 1 - \cos x$$



Excerpts from Anglin, W.S. (1994). *Mathematics: A Concise History*. New York, NY. Springer-Verlag.

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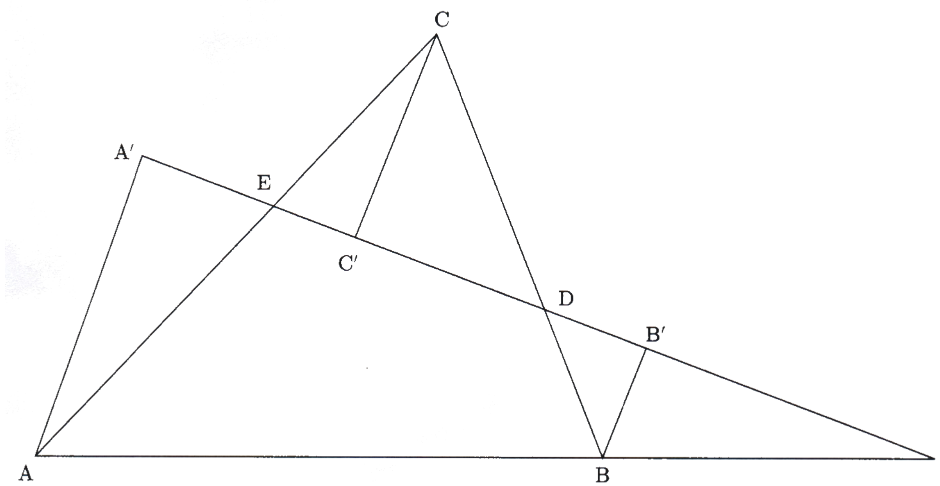
Menelaus of Alexandria (100 A.D.) founded spherical trigonometry. He is known for the following theorem, found, for the first time, in his *Spherica*.

> Let ABC be a triangle. Suppose D is on the line through B and C, E is on the line through A and C, and F is on the line through A and B. If exactly none or two of the points D, E, and F are on the sides of the triangle then: D, E, and F are collinear if and only if $BD \times CE \times AF = CD \times AE \times BF$.

Proof: Suppose D, E, and F are collinear in line z. We may suppose z does not pass through A, B, or C. Let A', B', and C' be points in z such that AA', BB', and CC' are all perpendicular to z. Then

$$\frac{BD}{CD} = \frac{BB'}{CC'}$$



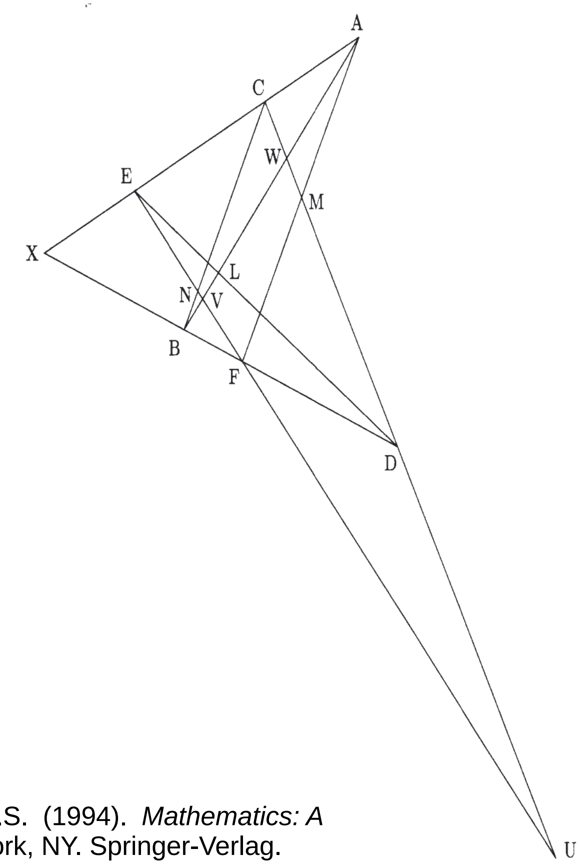
59

In 320 A.D., the Roman Empire had its first Christian emperor, Constantine. In Alexandria, Athanasius was defending the divinity of Jesus against Arius, who thought that Jesus was merely a special kind of human. Also in Alexandria, Pappus was writing his encyclopaedic *Collection.* The school of mathematics had declined, and Pappus was its last lone genius.

The following 'Theorem of Pappus' (actually due to Euclid) is Proposition 139 in Book VII of the *Collection.* It is more important than Pappus realised. It expresses the commutativity of multiplication. It is fundamental in projective geometry. Hilbert used it as a key theorem in his presentation of Euclidean geometry.

The Theorem of Pappus

Suppose we have a straight line with points X, E, C, and A on it (in that order) and another straight line with points X, B, F, and D on it (in that order), meeting the first line in X. Suppose ED meets AB in L, and EF meets BC in N, and CD meets AF in M. Then L, M, and N are collinear.



Excerpts from Anglin, W.S. (1994). *Mathematics: A Concise History*. New York, NY. Springer-Verlag.

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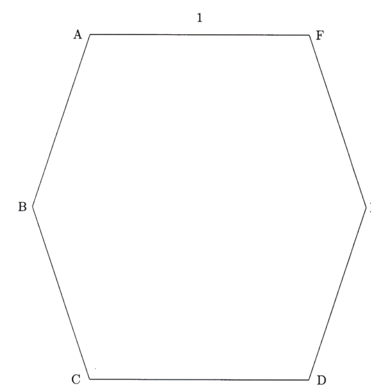
The ancient Greeks were fascinated by the 5 Platonic solids. Without the help of trigonometry or calculus, they managed to prove all the basic properties of these solids. The final book in Euclid's *Elements* is devoted to them. For each of the five regular polyhedra, Euclid calculates the ratio of its side to the radius of the sphere that circumscribes it.

For example, if one cuts an icosahedron in half, cutting along an edge AF (of length, say, 1), the resulting cross-section is a hexagon $ABCDEF$ with CD the edge of the icosahedron opposite AF (see the figure below). AC and DF are diagonals in regular pentagons formed by the sides of the icosahedron. (You may have to construct an icosahedron out of, say, cardboard in order to see this.) Thus if $AF = CD = 1$, $AC = DF = \phi$, where ϕ is the 'golden ratio' $\frac{1+\sqrt{5}}{2}$. (See Exercises 4, number 3.)

The diameter of the circumscribing sphere is CF, which is the hypotenuse of the right triangle with sides CD and DF. Thus $CF^2 = 1^2 + \phi^2$, and hence the radius of the circumscribing sphere is $\frac{\sqrt{1+\phi^2}}{2}$.

Note that ϕ is also the ratio of the side to the base in a triangle that is one of the points of the 5-pointed Pythagorean star. For Leonardo da Vinci (1452–1519), this 'golden ratio' was a mark of beauty. The icosahedron or dodecahedron is beautiful partly because it expresses this ratio.



Towards the end of this era, Greek mathematics degenerated into mere commentaries and riddle solving. In 529 Emperor Justinian closed Plato's Academy, apparently because it opposed the Christian revelation, and the few remaining scholars went to Persia. In 641 A.D., Alexandria fell to the Arabs, who burned the famous library. This event may be taken to mark the final end of ancient Greek mathematics.