

To many people, mathematics—unchanging, reliable, dusty with age—has an aura of authority and rests on a firm foundation of pure logic. It promises certainty. Schoolchildren learn strict, seemingly infallible rules, ranging from the mechanics of addition to the intricacies of factoring algebraic expressions. Engineers routinely calculate specifications, counting on reference volumes filled with mathematical formulas. Stockbrokers use the logic built into elaborate computer programs to help make decisions about when to buy and sell.

But behind the apparently stolid, pristine, immutable public face of mathematics lies the exciting, turbulent, ever-changing world of mathematical research. Just as physics and other sciences go through episodes of both revolution and evolution, mathematics, too, changes and grows, not only in the way it is applied but also in its fundamental structure. New ideas are introduced; intriguing connections between old ideas are discovered. Chance observations and informed guesses develop into whole new fields of inquiry.

The territory of mathematics can be divided into three large chunks. The first, called algebra, involves the study of number systems. In general, an algebra consists of a number of mathematical entities (such as integers, matrices, vectors, or sets) and operations (such as addition or multiplication) with formal rules expressing the relationships between the mathematical entities. It includes, for example, the rules needed for adding, subtracting, multiplying, and dividing the ones and zeros (or binary digits) that zip through a computer's mind and memory.

Number systems and the operations that are performed within them can be classified in much the same way that animals can be divided into families and species. Just as zoologists can place cats in the family of mammals, mathematicians have places for algebraic systems that obey certain rules. One particularly important and useful category is the "group," which shows up in all branches of mathematics and in crystallography, particle physics, and other sciences.

Analysis, the second major piece of the mathematical continent, concerns functions. Functions express relationships. Simply put, a function is any rule that assigns a fixed output to a given input. For example, if the function is squaring, the number 3 is paired with the number 9, the number 7 with the number 49. The development of calculus, discovered independently in the seventeenth century by Isaac Newton and Gottfried Leibniz, is one of the central achievements of analysis.

The third region, geometry, is the study of the properties of shapes and spaces. Most people are familiar with the rigid forms of euclidean geometry—squares and cubes, circles and spheres, congruent triangles and parallel lines. But geometries can take on many different guises, reaching into higher dimensions and obeying various rules. Topology focuses on geometrical features that remain unchanged after twisting, stretching, and other deformations are imposed on a geometric space. Problems such as coloring maps, distinguishing knots, and classifying surfaces, or manifolds, not just in one, two, and three dimensions, but in higher ones as well, all fall within topology.

Lying a short distance offshore from the mathematical mainland are the islands of number theory and set theory. Number theory, at one time considered the purest of pure mathematics, is simply the study of whole numbers, including prime numbers. This abstract field, once a playground for a few mathematicians fascinated by the curious properties of numbers, now has considerable practical value. Identifying primes and finding the prime factors of a composite number play a crucial role in many cryptographic schemes—systems designed to keep secrets secret.

Somewhere near the mathematical continent lie the burgeoning landmasses of statistics and computer science. Both have close ties with mathematics, and the links are becoming increasingly important.

Computer science, for example, can be thought of as the study of *algorithms*: the methods or procedures used to solve given classes of problems. In cooking, a recipe is the algorithm that guides a cook in transforming a motley collection of ingredients into a scrumptious cake. Mathematicians also need recipes. In classical geometry, the ancient Greeks devised a slew of procedures, employing only ruler and compass as tools, for performing a variety of geometric feats, including the bisection of angles and the drawing of regular figures such as hexagons. Later mathematicians spent much of their time looking for algorithms for efficiently computing π , finding logarithms, identifying primes, and performing countless other mathematical tasks.

Today's explosive growth in computer use adds urgency to the investigation of algorithms. Because computers operate on the basis of a small, built-in set of operations, the programs that instruct them, which are essentially algorithms, must be made as efficient and reliable as possible. The mathematical analysis of algorithms has spawned the field called computational complexity.

Although mathematical methods play important roles in statistics and computer science, the focus in these fields is on accomplishing particular goals in an efficient, practical manner. In contrast, pure mathematics delves more deeply into the existence and nature of mathematical objects, even when these objects can't be computed or constructed.

Mathematical research in modern times is an extensive enterprise. Because thousands of mathematicians publish hundreds of thousands of pages of new mathematical findings every year, most mathematicians find it difficult to keep up with what's happening across the field. Most of the time, they have to be content with being up-to-date in only a narrow furrow in the field. Consequently, mathematics tends to stay tightly packed into segregated compartments.

Nevertheless, many of the most striking mathematical results of the last decade are notions developed in one field that turn out to be a key element in solving outstanding problems in another, seemingly unrelated field. For example, in 1984, Dutch mathematician Hendrik Lenstra, out of curiosity, decided to study elliptic curves: equations of the form $y^2 = x^3 + ax + b$, where values for a and b are chosen arbitrarily. To his surprise, Lenstra serendipitously noticed a connection between elliptic curves and the age-old problem of determining the factors of an integer. His insight led to a new, speedier method of factoring large numbers. In another case of mathematical transfer, mathematician Vaughan F. R. Jones discovered a connection between operator algebras and knot theory, two topics that didn't have an obvious link. The result was an improved method for distinguishing different types of knots. British mathematician Simon Donaldson took the theory of Yang-Mills fields, which plays a role in the study of electromagnetic effects, out of physics and brought it to bear on a problem in topology. The result was the startling and unexpected discovery of a particularly bizarre geometry in four-dimensional space. All these examples attest to the essential unity of mathematics.

Even a brief glance at modern mathematical research reveals a dynamic enterprise of provocative ideas and questions. Far from being a domain of largely settled questions, mathematics is truly a wilderness. The well-mapped settlements lie few and far between, scattered across the continent and linked by a still skimpy network of highways and trails, some better traveled than others.