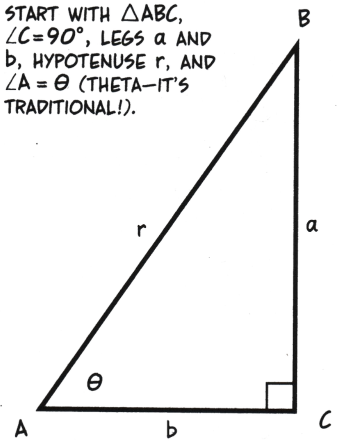
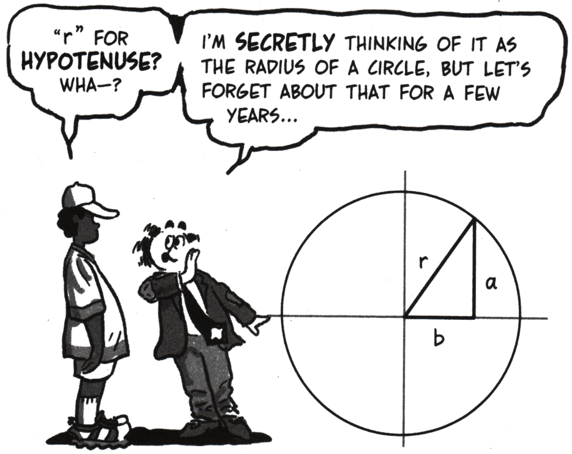
Sine and Cosine

START WITH $\triangle ABC$, $\angle C = 90°$, LEGS a AND b, HYPOTENUSE r, AND $\angle A = \theta$ (THETA—IT'S TRADITIONAL!).





Definitions

THE **SINE** OF θ, WRITTEN $\sin\theta$, IS THE RATIO a/r.

$$\sin\theta = \frac{\text{SIDE OPPOSITE } \theta}{\text{HYPOTENUSE}}$$

THE **COSINE** OF θ, WRITTEN $\cos\theta$, IS THE RATIO b/r.

$$\cos\theta = \frac{\text{SIDE ADJACENT } \theta}{\text{HYPOTENUSE}}$$

MATHEMATICIANS HAVE FOUND WAYS TO CALCULATE THESE RATIOS TO VERY HIGH ACCURACY. MY PHONE'S CALCULATOR GIVES THEM TO **15** DECIMAL PLACES. FOR THE SINE, FIRST PUNCH IN THE ANGLE MEASURE, THEN THE "SIN" KEY. FOR THE COSINE, HIT "COS."

$\sin 25° = 0.422618261740699...$

$\cos 89° = 0.017452406437283...$

$\sin 30° = 0.5 \quad (=\tfrac{1}{2})$

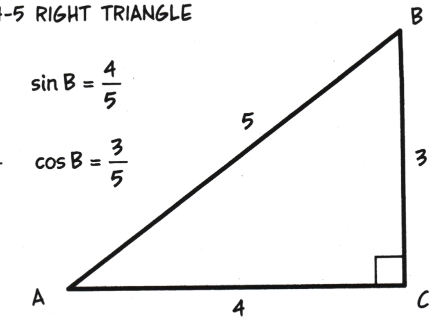
$\cos 45° = \dfrac{\sqrt{2}}{2} = 0.707106781186547...$

Examples

1. IN A 3-4-5 RIGHT TRIANGLE

$\sin A = \frac{3}{5}$ $\quad \sin B = \frac{4}{5}$

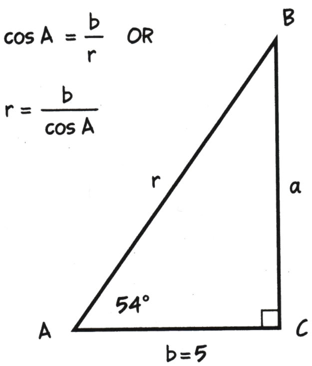
$\cos A = \frac{4}{5}$ $\quad \cos B = \frac{3}{5}$



2. THIS RIGHT TRIANGLE HAS ONE KNOWN ANGLE, $\angle A = 54°$, AND ONE KNOWN SIDE, $b = 5$. BY DEFINITION,

$\cos A = \dfrac{b}{r}$ OR

$r = \dfrac{b}{\cos A}$



WE FIND $\cos 54°$ ON THE CALCULATOR

$\cos 54° \approx 0.587785$

SO

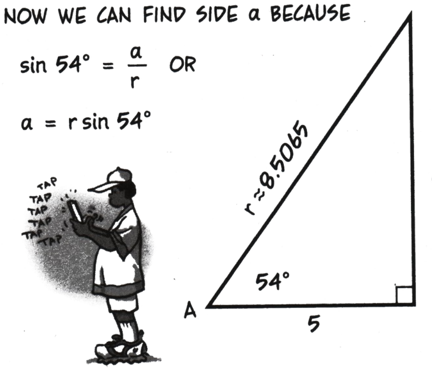
$r \approx \dfrac{5}{0.587785}$

$\approx \mathbf{8.5065}$

NOW WE CAN FIND SIDE a BECAUSE

$\sin 54° = \dfrac{a}{r}$ OR

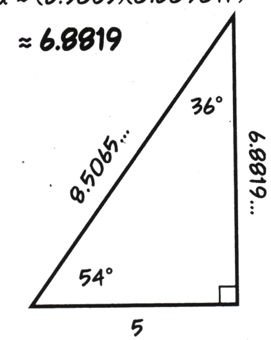
$a = r \sin 54°$



AGAIN USE THE CALCULATOR TO FIND $\sin 54° \approx 0.809017$ AND PLUG THAT IN:

$a \approx (8.5065)(0.809017)$

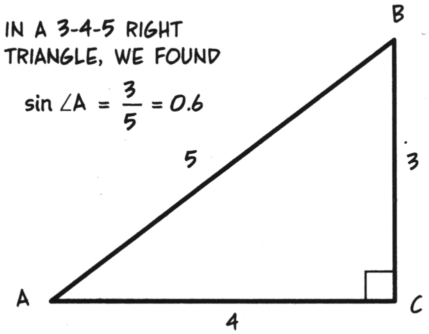
$\approx \mathbf{6.8819}$



IN RIGHT TRIANGLES, WE CAN ALSO CALCULATE ANGLES BASED ON SIDES.

IN A 3-4-5 RIGHT TRIANGLE, WE FOUND

$$\sin \angle A = \frac{3}{5} = 0.6$$

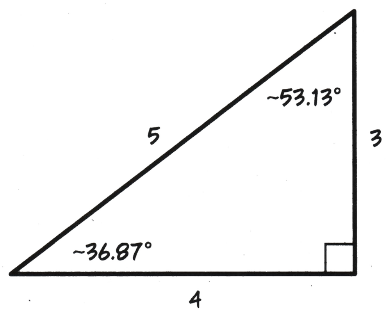


AND THEN? HOW DO WE FIND $\angle A$ FROM ITS SINE? AGAIN, BREAK OUT THE CALCULATOR.

1. ENTER 0.6.
2. PRESS THE "INV" OR "2ND FUNCTION" KEY.
3. PRESS $\sin^{-1}$.
4. READ **36.8698...**

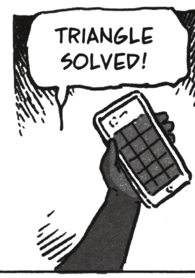
$$\angle A = \mathbf{36.8698...^\circ}$$

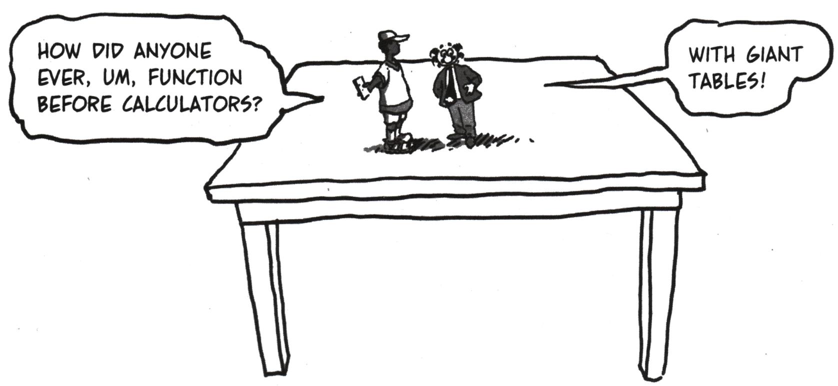
THIS $\sin^{-1}$ IS CALLED THE **INVERSE SINE** OR **ARCSINE**. FEED IT A NUMBER BETWEEN ZERO AND 1, AND IT RETURNS **THE ANGLE WITH THAT SINE**. IN THIS CASE, THE ANGLE HAVING SINE 0.6 IS AROUND 36.87°. THERE IS ALSO AN INVERSE COSINE KEY, $\cos^{-1}$.



$\angle B$, OF COURSE, IS THE COMPLEMENT OF 36.87°.

$$\angle B \approx 90° - 36.87$$
$$\approx \mathbf{53.13°}$$

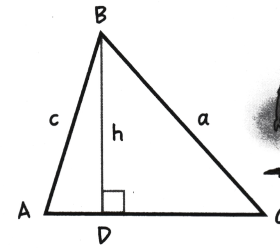




WE SOLVE OTHER TRIANGLES BY DIVIDING THEM INTO TWO RIGHT TRIANGLES. IF THE ALTITUDE BD = h, THEN

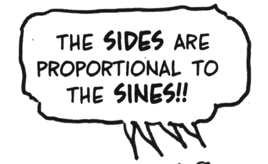
$$\frac{h}{a} = \sin C \qquad h = a \sin C$$

$$\frac{h}{c} = \sin A \qquad h = c \sin A$$



Theorem 20-1 (THE LAW OF SINES). IN A TRIANGLE $\triangle ABC$ WITH THREE ACUTE ANGLES

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



Proof. WE HAVE TWO DIFFERENT EXPRESSIONS FOR THE ALTITUDE h, WHICH MUST BE EQUAL.

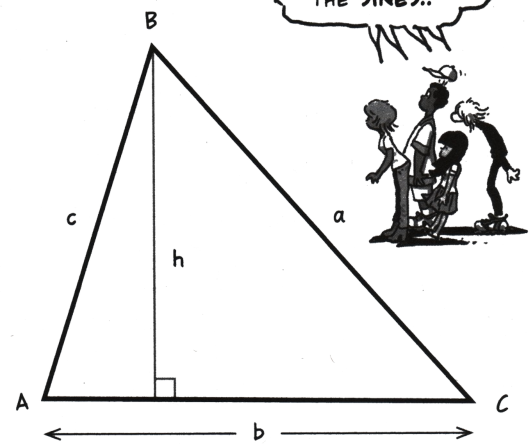
$$c \sin A = a \sin C$$

DIVIDING BY ac GIVES

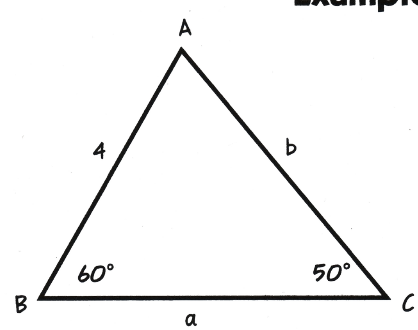
$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

SIMILARLY, USING THE ALTITUDE FROM A IN THE SAME WAY GIVES

$$\frac{\sin C}{c} = \frac{\sin B}{b} \blacksquare$$



Example: SUPPOSE WE KNOW TWO ANGLES AND A SIDE.



PLUGGING IN GIVES

$$\frac{\sin C}{c} \approx \frac{\sin 50°}{4} \approx \frac{0.766}{4} \approx 0.1915$$

BY THE LAW OF SINES,

$$0.1915 \approx \frac{\sin B}{b} = \frac{\sin 60°}{b} \approx \frac{0.86}{b}, \text{ SO}$$

$$b \approx \frac{0.86}{0.1915} \approx \mathbf{4.491}$$

SINCE $\angle A = 70°$, WE CAN FIND $a \approx 4.907$ IN THE SAME WAY.

Excerpts from Gonick, Larry (2024). *The Cartoon Guide to Geometry*. New York, NY. Harper Collins.