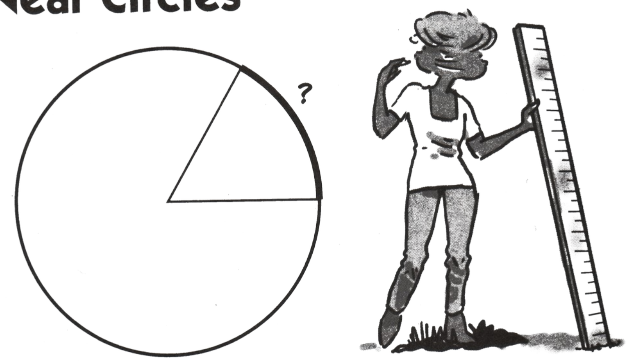
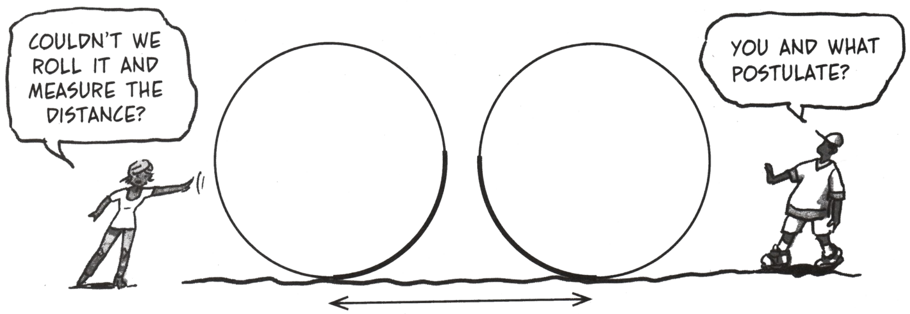
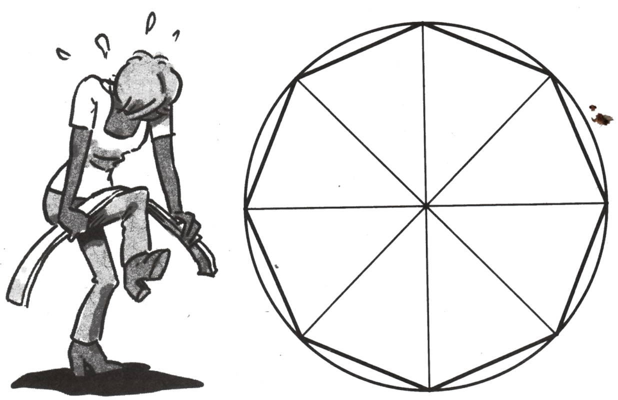
Polygons as Near Circles

IN ALL THE TALK ABOUT LENGTH AND AREA IN THIS BOOK, WE'VE NEVER MENTIONED CIRCLES. WHAT'S THE **LENGTH** OF AN **ARC?**

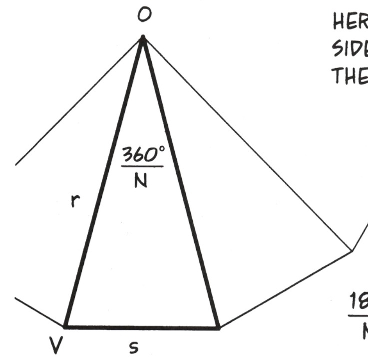


THE QUESTION REMAINS UNANSWERED BECAUSE THE RULER POSTULATE APPLIES ONLY TO STRAIGHT LINES. OUR RULERS ARE STRAIGHT AND RIGID, NOT FLOPPY LIKE TAPE MEASURES. TO US, THE LENGTH OF A CURVE HAS NO MEANING—YET.



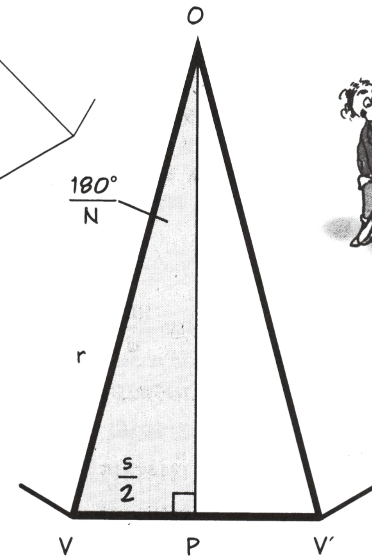


POLYGONS, ON THE OTHER HAND, HAVE STRAIGHT SIDES. WE CAN DIVIDE THEM INTO TRIANGLES AND USE ALL THE TOOLS AT OUR DISPOSAL, INCLUDING THE TRIG FUNCTIONS.



HERE'S A WEDGE OF A REGULAR N-SIDED POLYGON WITH SIDE s. O IS THE CENTER; V IS ANY VERTEX; AND r IS THE RADIUS OV. THE TRIANGLE'S APEX ANGLE IS 360°/N.

AN ALTITUDE OP OF $\triangle OVV'$ BISECTS THE CENTRAL ANGLE AND THE SIDE VV′ (AS $\triangle OVV'$ IS ISOSCELES). THE SINE FUNCTION RELATES THE RADIUS, THE HALF-ANGLE, AND THE HALF-SIDE.



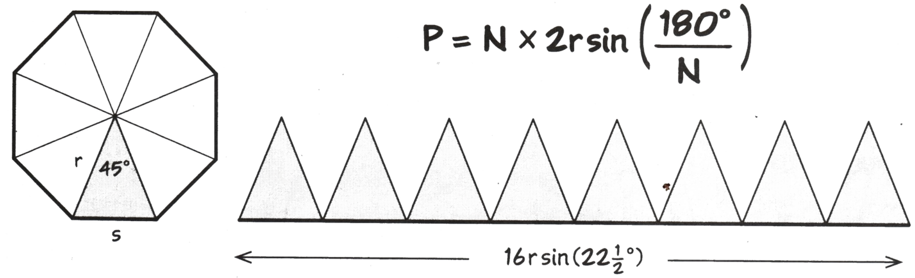
$$\frac{\left(\frac{s}{2}\right)}{r} = \sin\left(\frac{1}{2}\cdot\frac{360°}{N}\right)$$

$$\frac{s}{2r} = \sin\left(\frac{180°}{N}\right)$$

$$s = 2r\sin\left(\frac{180°}{N}\right)$$

THE POLYGON'S **PERIMETER**—THE DISTANCE **P** ALL THE WAY AROUND, OR THE SUM OF ALL THE SIDES—IS **N × s,** AS ALL THE SIDES ARE THE SAME LENGTH.

$$P = N \times 2r\sin\left(\frac{180°}{N}\right)$$



FOR INSTANCE, IN A REGULAR OCTAGON, $\theta = 45°$ (360°/8), $\theta/2 = 22.5°$, AND FROM LAST CHAPTER'S EXERCISES,

$$\sin 22.5° = \left(\frac{\sqrt{2-\sqrt{2}}}{2}\right)$$

SO ONE SIDE IS

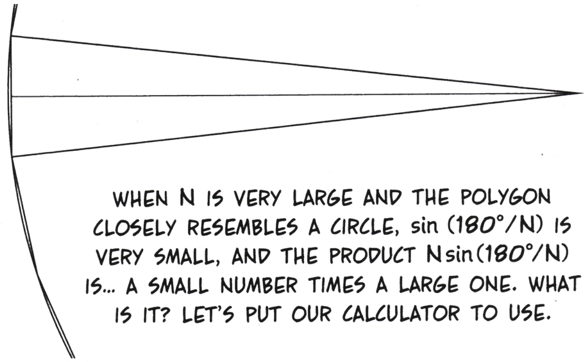
$$2r\cdot\tfrac{1}{2}\left(\sqrt{2-\sqrt{2}}\right) \approx r(0.7653686)$$

AND THE PERIMETER IS APPROXIMATELY

$$8r(0.7653686) = \mathbf{6.122949r}$$

LET'S GROUP TERMS OF THE PERIMETER FORMULA INTO THOSE THAT DEPEND ON N AND THOSE THAT DON'T.

$$P = (2r)\underbrace{\left(N \sin \frac{180^\circ}{N}\right)}_{\text{TERMS DEPENDING ON N}}$$



WHEN N IS VERY LARGE AND THE POLYGON CLOSELY RESEMBLES A CIRCLE, $\sin(180°/N)$ IS VERY SMALL, AND THE PRODUCT $N\sin(180°/N)$ IS... A SMALL NUMBER TIMES A LARGE ONE. WHAT IS IT? LET'S PUT OUR CALCULATOR TO USE.

N	$\frac{180^\circ}{N}$	$\sin\frac{180^\circ}{N}$	$N\sin\frac{180^\circ}{N}$
12	15°	0.258819...	3.1058285...
50	3.6°	0.06279...	3.139525...
180	1°	0.01745...	3.14143315...
1,800	0.1°	0.001745...	3.14159105...
180,000	0.001°	0.000017453...	3.14159265...

THE NUMBERS IN THE LAST COLUMN APPROACH A **LIMIT** SLIGHTLY MORE THAN 3.14159265, A NUMBER FAMILIAR TO MANY OF YOU AS

π

GREEK LETTER PI

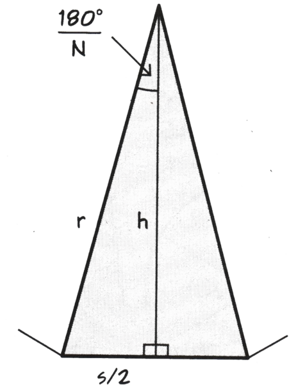
AND WE CONCLUDE, BY SOME MYSTERIOUS PROCESS OF "PASSING TO THE LIMIT," THAT **C, THE CIRCUMFERENCE OF A CIRCLE OF RADIUS r,** IS

WHAT ABOUT **AREA?** EACH TRIANGULAR WEDGE HAS AREA $\mathcal{A}_w = \frac{1}{2}hs$, WHERE h IS AN ALTITUDE. BUT

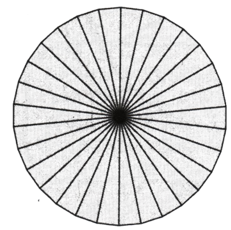
$$\frac{s}{2} = r \sin \frac{180^\circ}{N} \qquad h = r \cos \frac{180^\circ}{N}$$

SO

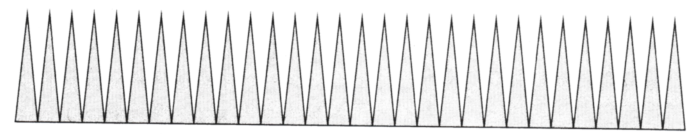
$$\mathcal{A}_w = \frac{hs}{2} = r^2 \sin \frac{180^\circ}{N} \cos \frac{180^\circ}{N}$$



AND THE POLYGON'S **TOTAL AREA $\mathcal{A}$** IS N OF THESE:

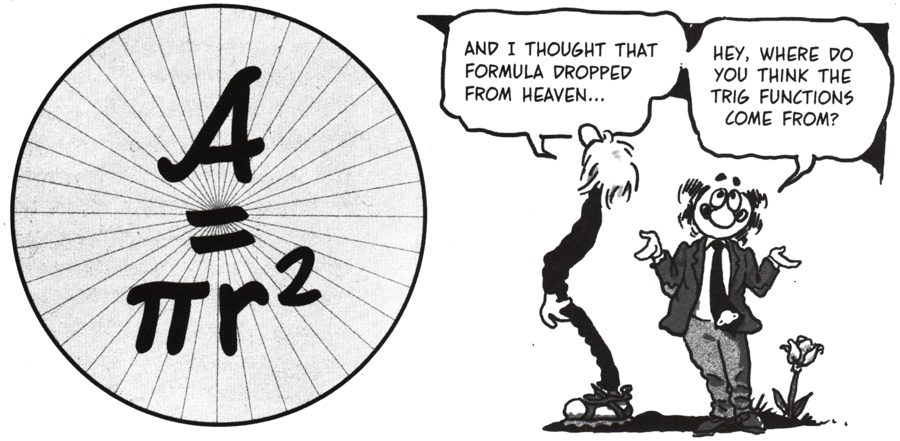


$$\mathcal{A} = N\mathcal{A}_w = r^2 N \sin \frac{180^\circ}{N} \cos \frac{180^\circ}{N}$$



WHEN N IS VERY LARGE, $N\sin(180°/N)$ APPROACHES π, AS WE SAW. MEANWHILE $\cos(180°/N)$ NEARS 1 BECAUSE h BECOMES "ALL BUT EQUAL TO" r. IN SHORT—

$$\mathcal{A} = r^2 \underbrace{N \sin \frac{180^\circ}{N}}_{\pi} \underbrace{\cos \frac{180^\circ}{N}}_{1}$$



Excerpts from Gonick, Larry (2024). *The Cartoon Guide to Geometry*. New York, NY. Harper Collins.