The **real numbers** are already familiar to you; they are just those numbers ordinarily used in most measurements. The mass, speed, temperature, and charge of a body are measured by real numbers. Real numbers can be represented by **terminating** or **nonterminating** decimal expansions. Any terminating decimal can be written in nonterminating form by adding zeros:

$$\tfrac{3}{8} = 0.375 = 0.375000000\ldots.$$

Any **repeating** nonterminating decimal, such as

$$\tfrac{7}{22} = 0.31818181818\ldots,$$

represents a **rational** number, one that is the quotient of two integers. Conversely, every rational number is represented by a repeating decimal expansion (as displayed above). The decimal expansion of an **irrational** number (one that is not rational), such as

$$\sqrt{2} = 1.414213562\ldots$$

or

$$\pi = 3.141592653589793\ldots,$$

is both nonterminating and nonrepeating.

The geometric representation of real numbers as points on the **real line** $\mathscr{R}$ should also be familiar to you. Each real number is represented by precisely one point of $\mathscr{R}$, and each point of $\mathscr{R}$ represents precisely one real number. By convention, the positive numbers lie to the right of zero and the negative numbers to its left, as indicated in Fig. 1.1.

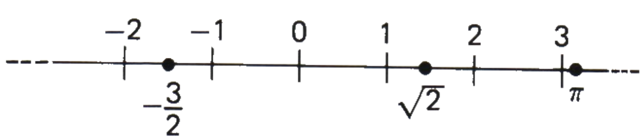
If the real number a lies (on $\mathscr{R}$) to the left of the number b, then we write $a < b$ (or $b > a$) and say that a is *less than* b and that b is *greater than* a. The number a is *positive* if $a > 0$; if $a < 0$, then a is said to be *negative*. The following properties of inequalities of real numbers are fundamental and are often used.

$$\begin{aligned} &\text{If } a < b \text{ and } b < c, \text{ then } a < c.\\ &\text{If } a < b \text{ then } a + c < b + c.\\ &\text{If } a < b \text{ and } c > 0, \text{ then } ac < bc.\\ &\text{If } a < b \text{ and } c < 0, \text{ then } ac > bc.\end{aligned} \tag{1}$$

The last two statements mean that an inequality is preserved when its members are multiplied by a *positive* number but *reversed* when they are multiplied by a *negative* number.

In the problems at the end of this section, we ask you to deduce from the properties in (1) that the sum of two positive numbers is positive, while the sum of two negative numbers is negative; the product of two positive numbers *or* of two negative numbers is positive, while the product of a positive number and a negative number is negative. Moreover,

$$a > b \quad \text{if and only if} \quad a - b > 0. \tag{2}$$



1.1 The real line $\mathscr{R}$

ABSOLUTE VALUE

The (nonnegative) distance along the real line between zero and the real number a is the **absolute value** of a, written $|a|$. Equivalently,

$$|a| = \begin{cases} a & \text{if} \quad a \geqq 0;\\ -a & \text{if} \quad a < 0.\end{cases} \tag{3}$$

The notation $a \geqq 0$ means that a is *either* greater than zero *or* equal to zero. Note that (3) implies that $|a| \geqq 0$ for every real number a, while $|a| = 0$ if and only if $a = 0$. For example,

$$|4| = 4, \qquad |-3| = 3, \qquad |0| = 0, \quad \text{and} \quad |\sqrt{2} - 2| = 2 - \sqrt{2},$$

the latter being true because $2 > \sqrt{2}$. Thus $\sqrt{2} - 2 < 0$, and hence

$$|\sqrt{2} - 2| = -(\sqrt{2} - 2) = 2 - \sqrt{2}.$$

The following properties of absolute values are frequently used.

$$\begin{aligned} &|a| = |-a| = \sqrt{a^2} \geqq 0,\\ &|ab| = |a|\,|b|,\\ &-|a| \leqq a \leqq |a|,\\ &|a| < b \quad \text{if and only if} \quad -b < a < b,\\ &|a| = b \geqq 0 \quad \text{if and only if} \quad a = b \text{ or } a = -b, \qquad \text{and}\\ &|a| > b \geqq 0 \quad \text{if and only if} \quad a < -b \text{ or } a > b.\end{aligned} \tag{4}$$

The **distance** between the real numbers a and b is defined to be $|a - b|$ — or, if you prefer, $|b - a|$, which is equal to $|a - b|$. This distance is simply the length of the line segment of the real line $\mathscr{R}$ with endpoints a and b, as indicated in Fig. 1.2.

The properties of inequalities and of absolute values listed above imply the following important fact.

$$\boxed{\textbf{\textit{Triangle Inequality}} \quad |a + b| \leqq |a| + |b|.} \tag{5}$$

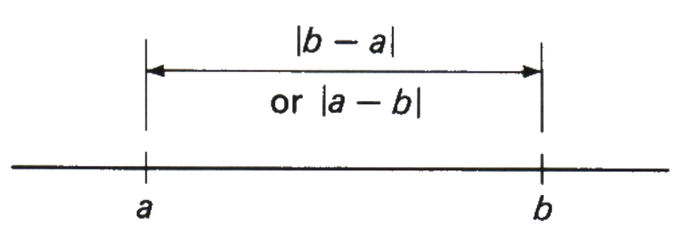
Proof We add the inequalities $-|a| \leqq a \leqq |a|$ and $-|b| \leqq b \leqq |b|$. This gives us

$$-(|a| + |b|) \leqq a + b \leqq |a| + |b|. \tag{6}$$

But the properties in (4) imply that

$$|c| \leqq d \quad \text{if and only if} \quad -d \leqq c \leqq d. \tag{7}$$

We set $c = a + b$ and $d = |a| + |b|$, and it follows that (5) and (6) are equivalent. ■



1.2 The distance between a and b

Excerpts from Edwards, C. and Penney, D. (1982). *Calculus and Analytic Geometry*. Athens, GA. Prentice Hall.

The key to the mathematical analysis of a geometric or scientific situation is often the recognition of relationships between the variables that describe the situation. Such a relationship may be a formula that expresses one variable as a function of another. For example, the area A of a circle of radius r is given by $A = \pi r^2$. The volume V and surface area S of a sphere of radius r are given by $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$, respectively. After t seconds a body that has been dropped from rest has fallen a distance $s = \frac{1}{2}gt^2$ feet and has speed $v = gt$ feet per second, where $g \approx 32$ ft/s^2 is gravitational acceleration. The volume V (in liters) of 3 g of carbon dioxide (CO_2) at 27°C is given in terms of its pressure p (in atmospheres) by $V = 1.68/p$. These are all examples of real-valued functions of a real variable.

Definition of a Function

A real-valued **function** f defined on a set D of real numbers is a rule that assigns to each number x in D exactly one real number $f(x)$.

The set D of all those numbers x for which $f(x)$ is defined is called the **domain** (or **domain of definition**) of the function f. The number $f(x)$, read "f of x," is called the **value** of the function f at the number or point x. The set of all values $y = f(x)$ is called the **range** of f. That is, the range of f is the set

$$\{y : y = f(x) \quad \text{for some } x \text{ in } D\}.$$

In this section we will be more concerned with the domain of a function than with its range.

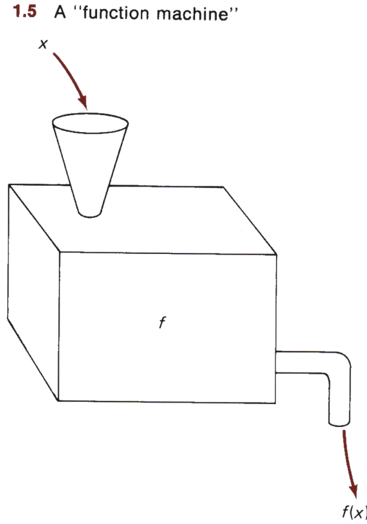
A function f with domain D and range R may be thought of as a machine that accepts a number x from D and then puts out (or displays or prints) the number $f(x)$, as indicated in Fig. 1.5. Such a machine is the familiar pocket calculator with a $\sqrt{\ }$ button. When a nonnegative number x is entered and this button is pressed, the calculator displays (an approximation to) the number $\sqrt{x}$. If such a calculator were not limited to eight- or ten-digit accuracy, it would be a perfect model for the square root function. In this case it is easy to see that the domain D is the set of all nonnegative real numbers and the range is the same (because $\sqrt{x}$ always denotes the *nonnegative* square root of x).

Often a function is described by means of a formula that specifies how to compute the number $f(x)$ in terms of the number x. For example, the formula

$$f(x) = x^2 + x - 3 \tag{1}$$

describes the rule of a function f having as its domain the entire real line $\mathscr{R}$. Some typical values of f are $f(-2) = -1$, $f(0) = -3$, and $f(3) = 9$.

1.5 A "function machine"



Excerpts from Edwards, C. and Penney, D. (1982). *Calculus and Analytic Geometry*. Athens, GA. Prentice Hall.

Imagine the flat, featureless two-dimensional plane of Euclid's geometry. Install a copy of the real number line $\mathscr{R}$, with the line horizontal and the positive numbers to the right, as in Fig. 1.10. Add another copy of $\mathscr{R}$ perpendicular to the first, with the two lines crossing where zero is located on each. The second line should have the positive numbers above and the negative numbers below the first line. The first line is called the ***x*-axis** and the second is the ***y*-axis.**

With these added features, we call the plane the **coordinate plane,** because it is now possible to locate any point there by a pair of numbers called the *coordinates of the point*. If P is a point in the plane, draw perpendiculars from P to the coordinate axes, as shown in Fig. 1.11. One perpendicular meets the x-axis in the ***x*-coordinate** of P, called x_1. The other meets the y-axis in the ***y*-coordinate** y_1 of P. The pair of numbers (x_1, y_1) is called the **coordinate pair** for P, or simply the **coordinates** of P. In compact notation, we speak of "the point $P(x_1, y_1)$." The numbers x_1 and y_1 also are called the **abscissa** and **ordinate,** respectively, of the point P.

This coordinate system is called the **rectangular coordinate system,** or sometimes the **Cartesian coordinate system** (because its use in geometry was popularized, beginning in the 1630s, by the French mathematician and philosopher René Descartes (1596–1650)). The plane, thus coordinatized, is sometimes denoted by $\mathscr{R}^2$ because of the two copies of $\mathscr{R}$ that are used and is sometimes called the **Cartesian plane.**

Rectangular coordinates are easy to use because $P(x_1, y_1)$ and $Q(x_2, y_2)$ denote the same point when and only when $x_1 = x_2$ and $y_1 = y_2$. Thus when you know that P and Q are different points, you may conclude that P and Q have different abscissas or different ordinates (or both).

The point of symmetry $(0, 0)$ where the coordinate axes cross is called the **origin.** The points on the x-axis all have coordinates of the form $(x, 0)$, and, while the *real number* x is not the same as the *geometric point* $(x, 0)$, there are situations in which it is useful to think of the two as the same. Similar remarks can be made about the points $(0, y)$ that constitute the y-axis.

The concept of distance in the plane is based on the **Pythagorean theorem:** If ABC is a right triangle with right angle C and hypotenuse c (see Fig. 1.12), then

$$c^2 = a^2 + b^2. \tag{1}$$

The converse of the Pythagorean theorem is also true—that is, if the three sides of a given triangle satisfy the Pythagorean relation in (1), then the angle opposite side c is a right angle.

The *distance* $d(P_1, P_2)$ between the points P_1 and P_2 is (by definition) the length of the straight line segment joining P_1 and P_2. The following formula gives $d(P_1, P_2)$ in terms of the coordinates of the two points.

Distance Formula

The **distance** between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

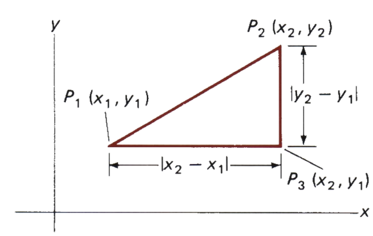
$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{2}$$

Proof If $x_1 \neq x_2$ and $y_1 \neq y_2$, then the formula in (2) follows from an application of the Pythagorean theorem. Use the right triangle with vertices P_1, P_2, and $P_3(x_2, y_1)$ shown in Fig. 1.13.

If $x_1 = x_2$, then P_1 and P_2 lie in a vertical line. In this case

$$d(P_1, P_2) = |y_2 - y_1| = \sqrt{(y_2 - y_1)^2}.$$

This agrees with the formula in (2) because $x_1 = x_2$. The remaining case, in which $y_1 = y_2$, is similar. ■



1.13 Use this triangle to deduce the distance formula.