61

Example 1.3.11 Prove that for each positive integer n, $|\sin(nx)| \le n(\sin(x))$ for $0 \le x \le \pi$. (Here, $|u|$ is the absolute value of the real number u: $|u| = u$ if $u \ge 0$ and $|u| = -u$ if $u < 0$.)

Proof For each positive integer n, define $S(n)$: $|\sin(nx)| \le n\sin(x)$ for $0 \le x \le \pi$.

Basis step. We must show that $|\sin(1 \cdot x)| = |\sin(x)| \le 1 \cdot \sin(x) = \sin(x)$ for $0 \le x \le \pi$. But for $0 \le x \le \pi$, $\sin(x) \ge 0$; hence $|\sin(x)| = \sin(x)$.

Inductive step. We show that for each positive integer k, $S(k)$ implies $S(k+1)$. Therefore, we assume that for $0 \le x \le \pi$, $|\sin(kx)| \le k\sin(x)$ and deduce that $|\sin(k+1)x| \le (k+1)\sin(x)$.

Although it is easy to obtain $(k+1)\sin(x)$ from $k \cdot \sin(x)$ and vice versa, it is not clear how to build $|\sin(k+1)x|$ from $|\sin(kx)|$. But the addition formula for the sine function, $\sin(u+v) = (\sin u)(\cos v) + (\cos u)(\sin v)$, can help us express $\sin(k+1)x$ in terms of $\sin(kx)$. Thus we shall attempt to manipulate the left side of $S(k+1)$ to the point where we can use statement $S(k)$. From the addition formula for the sine function,

$$\begin{aligned}|\sin(k+1)x| &= |\sin(kx+x)|\\ &= |\sin(kx)\cdot\cos(x) + \cos(kx)\cdot\sin(x)|\\ &\le |\sin(kx)\cdot\cos(x)| + |\cos(kx)\cdot\sin(x)|,\end{aligned}$$

since for all real numbers u and v, $|u+v| \le |u| + |v|$. (This is the so-called *triangle inequality*.) Moreover, since $|u \cdot v| = |u| \cdot |v|$ for all real u and v and since $|\cos(u)| \le 1$ for all real u, we have

$$\begin{aligned}|\sin(k+1)x| &\le |\sin(kx)\cdot\cos(x)| + |\cos(kx)\cdot\sin(x)|\\ &\le |\sin(kx)|\cdot|\cos(x)| + |\cos(kx)|\cdot|\sin(x)|\\ &\le |\sin(kx)| + |\sin(x)|.\end{aligned}$$

Recall that we are assuming that $S(k)$ holds. Thus $|\sin(kx)| \le k \cdot \sin(x)$. Also, since $0 \le x \le \pi$, $|\sin(x)| = \sin(x)$. It follows that

$$\begin{aligned}|\sin(k+1)x| &\le |\sin(kx)| + |\sin x|\\ &\le k\cdot\sin(x) + \sin(x) \le (k+1)\cdot\sin(x).\end{aligned}$$

Hence the inductive step is established.

By PMI, $|\sin(nx)| \le n \cdot \sin(x)$ for each positive integer n and each real number x such that $0 \le x \le \pi$. ■

62

One minor technical point arises on occasion. In some cases it is natural to label the statements on the list so that the first statement is $S(2)$, $S(5)$, or in general $S(k_0)$ for some integer $k_0 \ge 0$. For such cases we have a slightly more general version of PMI.

Principle of Mathematical Induction (Generalized). Let k_0 be a nonnegative integer and let $S(k_0), S(k_0+1), \ldots, S(n), \ldots$ be a list of statements, one for each integer $n \ge k_0$. Then every statement on the list is true if the following two conditions hold:

(i) $S(k_0)$ is true.
(ii) For each integer $k \ge k_0$, if $S(k)$ is true then $S(k+1)$ is true.

There is a different form of mathematical induction that possesses two significant virtues: It has the same effect as PMI, and it is applicable in many situations in which PMI cannot be so readily used. We refer to this form of mathematical induction as the Principle of Strong Induction (PSI), since the hypothesis in the inductive step of PSI is stronger than that of PMI.

Principle of Strong Induction. Let k_0 be a nonnegative integer and let $S(k_0), S(k_0+1), \ldots, S(n), \ldots$ be a list of statements, one for each positive integer greater than or equal to k_0. Then every statement on the list is true if the following two conditions hold:

(i) $S(k_0)$ is a true statement.
(ii) For each positive integer $k \ge k_0$, if $S(k_0), S(k_0+1), \ldots, S(k)$ are all true, then $S(k+1)$ is true.

First note that if PSI (i) and (ii) are established, then $S(k)$ is true for each positive integer $k \ge k_0$, for, if (i) is proved, then $S(k_0)$ is true. By (ii) applied in the case $k = k_0$, $S(k_0+1)$ is true. Since $S(k_0)$ and $S(k_0+1)$ are true, it follows from PSI (ii) that $S(k_0+2)$ is true. Since $S(k_0)$, $S(k_0+1)$, and $S(k_0+2)$ are true, $S(k_0+3)$ is true by PSI (ii). Again by PSI (ii), we deduce that $S(k_0+4)$ holds and so on. Thus PSI enables us to prove that $S(k)$ is true for each $k \ge k_0$.

Excerpts from Galovich, S. (1989). *Introduction to Mathematical Structures*. San Diego, CA. Harcourt Brace Jovanovich.

The important question remains: When should we use PSI instead of PMI? The basic idea in the inductive step of PMI is that $S(k+1)$ can be deduced from or reduced back to $S(k)$. This is precisely what occurred in Examples 1.3.8–1.3.11. In some cases, though, $S(k+1)$ cannot be readily deduced from $S(k)$, but can be derived from all of the previous statements $S(k_0), \ldots, S(k)$. In these cases we appeal to PSI in order to conclude that all the statements $S(k_0), \ldots, S(k), \ldots$ are true. Our next example illustrates this point.

Theorem 1.3.7 *Each integer greater than* 1 *is either prime or is a product of primes.*

Proof For an integer $n \geq 2$, define the statement $S(n)$: n is either prime or is a product of primes.

Basis step. $S(2)$ is true since 2 is a prime number.

Inductive step. Suppose statements $S(2), \ldots, S(n)$ are all true and show that $S(n+1)$ is true. Without doubt the integer $n+1$ either is prime or is not prime. If $n+1$ is a prime, then $S(n+1)$ is true. If $n+1$ is not prime, then $n+1 = a \cdot b$ where $a, b < n+1$; hence, $a, b > 1$. But by hypothesis $S(a)$ and $S(b)$ are both true. Thus, each of a and b either is prime or is a product of primes. Write $a = p_1 \cdot \cdots \cdot p_r$ and $b = q_1 \cdot \cdots \cdot q_s$ where $p_1, \ldots, p_r, q_1, \ldots, q_s$ are all prime. (For example, if $r = 1$, then a itself is prime.) Therefore,

$$n + 1 = a \cdot b = (p_1 \cdot \cdots \cdot p_r) \cdot (q_1 \cdot \cdots \cdot q_s)$$

and $a \cdot b$ is a product of primes.

By PSI, the proof is complete. ■

In the inductive step in the proof of Theorem 1.3.7, we used the fact that a and b are expressible as a product of primes to conclude that $n + 1 = a \cdot b$ is a product of primes. It is not at all obvious that this conclusion can be derived from the assumption that n is expressible as a product of primes.

Excerpts from Galovich, S. (1989). *Introduction to Mathematical Structures*. San Diego, CA. Harcourt Brace Jovanovich.

Case Analysis

The final method of proof to be discussed is proof by *case analysis*. In a case analysis proof of a given statement, the entire argument is divided into a collection of cases. Each of the cases is then resolved, thereby establishing the original statement.

For example, in proving some statement involving a single integer variable, call it x, one might consider two cases: (1) $x \geq 0$ and (2) $x < 0$. Or in proving a statement involving a real number x, one can analyze the following cases: (1) $x =$ integer, (2) $x =$ rational, (3) $x =$ arbitrary real.

We distinguish two types of case analysis proofs: *divide-and-conquer* and *bootstrap*. In a divide-and-conquer proof, the original problem is divided into a number of separate cases that are established (or conquered) independently of each other. In a bootstrap proof, the original problem is divided into a sequence of cases in which a given case after the first one is established with the use of some or all of the previous cases. We now illustrate each of these types of case analysis proofs. We begin with an example of a divide-and-conquer argument.

Example 1.3.12 (i) Find all real solutions to the inequality $|x-1| < |x-3|$.

Recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Thus,

$$|x-1| = \begin{cases} x-1 & \text{if } x-1 \geq 0, \text{ i.e., if } x \geq 1 \\ -(x-1) = 1-x & \text{if } x-1 < 0, \text{ i.e., if } x < 1 \end{cases}$$

and

$$|x-3| = \begin{cases} x-3 & \text{if } x \geq 3 \\ 3-x & \text{if } x < 3. \end{cases}$$

Thus we consider three cases: (1) $x \geq 3$, (2) $1 \leq x < 3$, (3) $x < 1$.