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If Paul Halmos, a well-known twentieth-century mathematician, is indeed correct, then the ability to solve mathematical problems, especially those that differ appreciably from any previously encountered exercise, is the foremost skill that a mathematician must possess. The typical person on the street probably believes that this ability is granted at birth to a chosen few and denied to the rest of the population. According to the popular folklore, either you are a mathematical genius or you can't do math at all.

On closer inspection, however, the simplicity of this viewpoint becomes apparent. Mathematical aptitudes and abilities vary over a seemingly continuous range. Some people prefer algebra to geometry in high school; others enjoy geometry far more than algebra. Some find high school mathematics easy and college mathematics difficult. Some are attracted more by "pure" mathematics than by "applied" mathematics. Others choose computational mathematics over "theorem-proving" mathematics. Moreover, as people study and practice mathematics, they generally do become more adept at learning new mathematics and more skillful at solving problems. In addition, many of us have encountered an inspiring mathematics teacher who aroused our interest and increased our performance in mathematics beyond what we thought possible. While no one can deny the existence of innate differences in mathematical ability among individuals, it is clear that environmental influences and personal characteristics are important in determining the mathematical performance of each individual.

Excerpts from Galovich, S. (1989). *Introduction to Mathematical Structures*. San Diego, CA. Harcourt Brace Jovanovich.

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Our question is, then: What can we as individuals do to improve our own mathematical performance? Certainly we can learn as much mathematics as possible. For example, the more we know, the more likely we are to be able to answer a particular question. But beyond that, how can we become better at solving problems, proving theorems, writing computer programs, building mathematical models of phenomena? In short, how can we become better at doing mathematics?

The usual answer to this question is: Practice. If you sit down and try to solve many mathematical problems or to prove many theorems, eventually you will become good at problem solving or theorem proving. The experience that you gain from failure, partial success, and success teaches you, perhaps slowly and painfully, how to do mathematics. But are "practice and experience" the only answers to these questions?

Our response to this question is a qualified No. Practice and the experience born of practice are necessary for the improvement of one's mathematical performance. But they are not sufficient to guarantee acceptable results. In fact, we assert that proficiency in mathematics requires

1. knowledge of subject matter,
2. awareness of strategies and tactics for solving problems and learning new material, and
3. ability to monitor and to control one's activities when doing mathematics.

Most mathematical textbooks have been concerned only with item 1. Indeed most of this text deals with the subject matter of a variety of topics. But in this section we focus on items 2 and 3. Our goal is to introduce and to discuss briefly some of the considerations involved in 2 and 3. Throughout the text frequent mention will be made of the ideas raised in this section.

What we present are some fundamental rules of thumb that might be of assistance in the problem-solving process. No one of the principles is guaranteed to work on any given problem. Nevertheless, they can serve to help a person discover a solution to a problem or a proof of a theorem. Principles of this sort—those that aim to help in the discovery of a solution but that are not certain to lead to a solution—are called *heuristic* principles. We now turn to a discussion of the heuristics, i.e., the rules of discovery, of problem solving.

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Our exposition is based on the work of George Polya and Alan Schoenfeld. Both Polya and Schoenfeld are professional mathematicians who have devoted considerable effort to the study of the activity of mathematical problem solving by humans. Polya's principal method was introspection: He carefully observed his own actions as he was solving mathematical problems, noting particularly the steps that enabled him to solve a difficult problem. In a well-known book, *How To Solve It*, Polya organized his findings into a scheme that divides the problem-solving processing into four parts:

1. Understand the Problem.
2. Devise a Plan.
3. Carry Out the Plan.
4. Look Back.

In contrast to Polya, Schoenfeld has carried out a number of controlled experiments with college students and professional mathematicians. His work has led him to modify Polya's scheme somewhat. In Schoenfeld's scheme there are three major steps:

1. Analysis.
2. Exploration.
3. Verification.

Excerpts from Galovich, S. (1989). *Introduction to Mathematical Structures*. San Diego, CA. Harcourt Brace Jovanovich.

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Focusing on the Unknown

When presented with a problem, the problem solver's first task is to understand the problem. Understanding a problem (or concept) involves several activities and stages; however, Polya's clear and forceful advice is to begin with the unknown. In Polya's chart (*How to Solve It*, pp. xvi–xvii) the first line begins with the question: What is the unknown? Focusing on the unknown provides coherence to the problem and directs the efforts of the problem solver toward a specific aim. In other words, if you know what you are looking for, then you are free to delve deeper into other aspects of the problem in order to obtain a more thorough understanding of it.

The unknown itself can take various forms. For example, it can be a specific mathematical object—a real number, a point in the plane, the point of intersection of two lines, a certain function, or a circle. In the latter case, for instance, to find the unknown circle, one must find its center (a point in the plane) and its radius (a positive real number). In other instances, the unknown is not so much a specific object as a task to perform or a goal to achieve: Prove that $\sqrt{2}$ is irrational, or find an efficient algorithm that will yield the solution to a given system of equations.

Whether looking for a specific object or a general task, by focusing on the unknown, the problem solver is forced to begin with the goal and to understand the meaning of all the terms in the unknown. What must be done to find a circle? What is an irrational number? As obvious as it might seem, the advice to focus on the unknown and to know the meaning of all the terms involved in the unknown (and the known, for that matter) is often overlooked.

As an exercise, take any mathematics book that you used in a previous course. Open the book to any section that you studied in that course and read over the exercises given at the end of the section. In each case identify the unknown and the meaning of the unknown. Repeat this procedure for several sections of the text and for the exercises in Section 1.3 of this chapter. Concentrate only on the identification and the meaning of the unknown. Ignore the techniques used in solving the problems. Does concentrating only on the unknown make the problems seem clearer and less forbidding now than when you first met them?