Problem Representation

Suppose that in a given problem you have isolated the unknown and the hypotheses or conditions. Suppose also that the meanings of all the terms involved in the problem are clear. What next? At this point the formulation of an adequate representation of the problem and its components will probably be most useful. By a representation of the problem, we mean a method of describing the problem that (i) is internally coherent or consistent, (ii) corresponds closely to the problem, and (iii) is closely connected with the knowledge of the problem solver about the items of the problem and related concepts. The representation might take any one of several forms. It might involve a picture, a graph, an algebraic equation or a system of algebraic equations, a differential equation or a system of differential equations, a vector diagram, or a mathematical system such as a vector space, a group, or a Boolean algebra. In summary, a representation of a problem amounts to a view of the problem that is sensible and relevant and can be manipulated in order to achieve a solution. Let us consider some examples of representations.

Perhaps the most familiar kind of representation is the pictorial or geometric representation of algebraic equations involving real numbers. We reconsider some previous examples to illustrate how a picture can be helpful.

Example 1.3.2 (revisited) If x is a real number and $x^2 - 4x + 6 = x$, then $x = 2$ or $x = 3$.

To represent this problem pictorially, recall that the equation $y = x^2 - 4x + 6$ has as its graph in the plane the parabola P with vertex at the point $(2, 2)$ and focus at $(2, 3)$. (The line $x = 2$ is the axis of symmetry of the parabola.) The graph of $y = x$ is the straight line L with slope 1 passing through the origin. The values of x for which $x^2 - 4x = x$ are precisely the first coordinates of the points at which the curves P and L intersect. The picture (Figure 1.4.1) makes the result appear extremely plausible. In fact, on the basis of the picture, we might

(i) guess that $x = 2$ and $x = 3$ are solutions to the original question,
(ii) check that $x = 2$ and $x = 3$ actually satisfy the equation,
(iii) show that no other solutions exist.

On the other hand, even if we do not use the picture (in this or any similar question) as a means of discovering solutions to the equation, we can use it as a check on a solution obtained by another method.

Finding Related Problems

The heart of the problem-solving process by Polya and Schoenfeld lies in the second stage, Devising a Plan or Exploration. At this point the would-be problem solver possesses a firm understanding of the elements of the problem, can construct an adequate representation of the problem, and perhaps can verify the result in special cases or can argue for the plausibility of a certain solution. In other words, the problem solver has a solid grip on the problem.

Even so, a solution might not be readily apparent. In fact, with most substantive problems, solutions rarely appear at the understanding stage of the problem-solving process. What then should one do in order to find a solution?

Polya's advice is clear: Find a related problem. The basic strategy is the following: Given a problem P,

1. find a related problem P',
2. solve problem P',
3. use the solution to P' or the method of solution of P' to solve P.

This plan faces several apparent obstacles. First, how does one actually find a related problem? Second, the related problem itself might be difficult to solve; it might be necessary to devise and solve a problem P'', related to P', and use the solution to P'' to solve P'. Finally, the transition from a solution to P' to a solution to P may be difficult to perform.

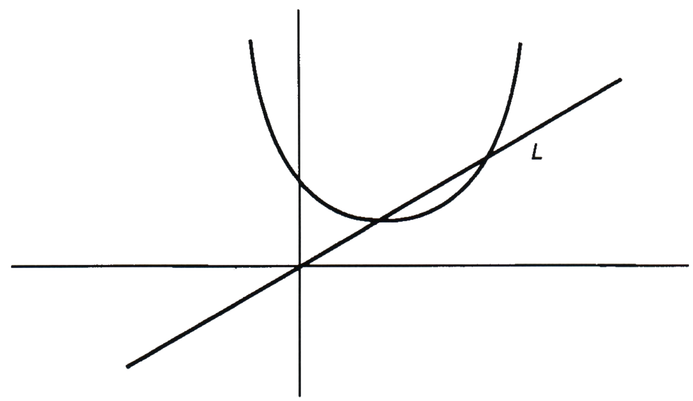

Figure 1.4.1

Excerpts from Galovich, S. (1989). *Introduction to Mathematical Structures*. San Diego, CA. Harcourt Brace Jovanovich.

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Generalization

A *generalization* of a given statement A is a statement B that includes A as a special case. In other words, the instances in which B applies include the instances in which A applies. Mathematicians are especially notorious for their tendency to generalize. One might reasonably question this practice. What are the advantages to generalization?

First, the process of generalizing often clarifies a problem. The essential features of a question are often obscured in a special case and are illuminated in the general situation. Details that seem important in a particular case are revealed to be less relevant when placed in a broader context. As a result a generalization of a statement is sometimes easier to handle than the original statement itself.

Second, through generalization we learn the limits of a given concept or method. By seeking to know the full extent to which a given statement is true, we develop a keener understanding of the concepts involved in it. If we can determine the instances in which a certain procedure applies, we can make maximum use of it while avoiding the pitfalls that arise from inappropriate use.

Finally, generalization can serve as a tool in research. Often a given idea can be generalized from a specific situation to a more encompassing setting in several ways. Each of these generalizations provides a starting point for further research and each such research effort creates a new body of mathematical knowledge. The process of generalization has the effect of extending and clarifying the concept that was initially considered.

To summarize in a few words, we generalize in order to know, to understand, to predict, and to create. The act of generalizing has one other notable benefit: It's fun.

Excerpts from Galovich, S. (1989). *Introduction to Mathematical Structures*. San Diego, CA. Harcourt Brace Jovanovich.

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Example 1.4.2 (revisited) In working out this example, we observed that a certain type of inequality is valid in three distinct cases. Are these three cases (involving two, three, or four variables) part of a more general pattern? Suppose we allow any fixed finite number of real numbers in the inequality: If n is a positive integer greater than 1 and $a_1, \ldots, a_n$ are real numbers strictly between 0 and 1, then $(1-a_1)(1-a_2)\cdots(1-a_n) > 1 - a_1 - a_2 - \cdots - a_n$. Note that when $n = 4$, the statement in Example 1.4.2 is recaptured.

Is this generalization true for all $n \geq 2$? Indeed, it is for $n = 2$, 3, or 4. This finding suggests a proof by mathematical induction. For each integer $n \geq 2$, let $S(n)$ be the statement: If $a_1, \ldots, a_n$ are real numbers strictly between 0 and 1, then $(1-a_1)(1-a_2)\cdots(1-a_n) > 1 - a_1 - a_2 - \cdots a_n$.

To prove that $S(n)$ holds for $n \geq 2$, we must show that

(i) $S(2)$ is true,
(ii) if $S(m)$ is true, then $S(m+1)$ is true.

We have already checked that $S(2)$ is true, so we consider the inductive step. We assume that $S(m)$ holds for some m and prove that $S(m+1)$ is valid.

We must prove that if $a_1, \ldots, a_{m+1}$ are real numbers between 0 and 1, then

$$(1-a_1)\cdots(1-a_{m+1}) > 1 - a_1 - \cdots - a_{m+1}.$$

From the inductive hypothesis we know that

$$(1-a_1)\cdot\cdots\cdot(1-a_m) > 1 - a_1 - \cdots - a_m.$$

Since $1 - a_{m+1} > 0$, we obtain from the latter inequality

$$\begin{aligned}(1-a_1)\cdot\cdots\cdot(1-a_m)(1-a_{m+1}) &> (1-a_1-\cdots-a_m)(1-a_{m+1})\\ &= 1 - a_1 - \cdots - a_m - a_{m+1}\\ &\quad + (a_1 + \cdots + a_m)\cdot a_{m+1}\\ &> 1 - a_1 - \cdots - a_m - a_{m+1}.\end{aligned}$$

Thus the inductive step is established and $S(n)$ is true for all $n \geq 2$.