Analogy

Given a problem A, an analogous problem B has the same "structure" as A but takes place in a different setting. For example, A might be a problem in three-dimensional geometry involving lines and planes while B is a problem in two-dimensional geometry having to do with points and lines, or A might be a problem involving a function of three real variables while B is a problem involving a function of one real variable. We consider in detail an example from algebra.

Example 1.4.2 Show that if a, b, c, d are real numbers strictly between 0 and 1, then $(1 - a)(1 - b)(1 - c)(1 - d) > 1 - a - b - c - d$.

Solution Our first impulse might be to multiply out $(1 - a)(1 - b)(1 - c)(1 - d)$ and show that the resulting mess is greater than $1 - a - b - c - d$. However, the extent of the mess involved encourages us to seek another approach.

Let us look for a similar or analogous problem involving fewer variables. Consider the statement: If a and b are real numbers strictly between 0 and 1, then $(1 - a)(1 - b) > 1 - a - b$. This statement, while having the same form as the original, involves only two variables. Moreover it is easy to check: Since $a \cdot b > 0$,

$$(1 - a)(1 - b) = 1 - a - b + a \cdot b > 1 - a - b.$$

We now add back a variable to obtain yet another analogous statement: If a, b, c are real numbers strictly between 0 and 1, then

$$(1 - a)(1 - b)(1 - c) > 1 - a - b - c.$$

To prove this statement, perhaps we can use the result in the two-variable case considered above:

$$(1 - a)(1 - b) > 1 - a - b.$$

Multiplying both sides by $(1 - c)$, we have

$$\begin{aligned}(1 - a)(1 - b)(1 - c) &> (1 - a - b)(1 - c)\\ &= 1 - a - b - c + (a + b)c > 1 - a - b - c\end{aligned}$$

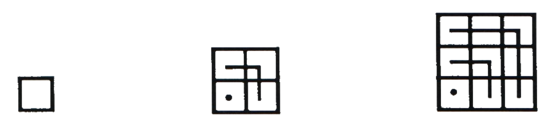
since $(a + b)c > 0$.

Now what about the original problem? As can be checked, this argument extends to provide a proof that $(1 - a)(1 - b)(1 - c)(1 - d) > 1 - a - b - c - d$ if a, b, c, d are strictly between 0 and 1. In fact, a pattern is emerging from these cases. As an exercise, try to describe that pattern and prove your description of it correct.

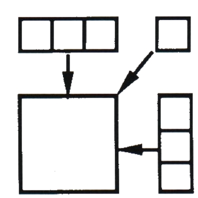
Looking Back

Polya's admonition that we look back over our work has two thrusts. First, he encourages us to check our work by applying various tests to the solution—use of data, symmetry, and alternative derivations. Second, he points us toward future problems by having us try to derive new results from the result just established. In other words, looking back has both a backward and a forward orientation. In this section we concentrate on the first feature. However, we shall comment briefly upon the second and shall illustrate both the backward and forward nature of checks throughout the text.

Example 1.4.4 The equality $1 + 3 + \cdots + (2n - 1) = n^2$ can be checked geometrically. Consider the squares



To obtain a 4×4 square from the 3×3 square, we give three squares to the top of the 3×3 square, three squares to the right side, and one square to the upper right.



Thus $4^2 - 3^2 = 3 + 3 + 1 = 7$. In general the $(n + 1) \times (n + 1)$ square is obtained by adding $n + n + 1 = 2n + 1$ squares to the $n \times n$ square.

Excerpts from Galovich, S. (1989). *Introduction to Mathematical Structures*. San Diego, CA. Harcourt Brace Jovanovich.

Overcoming Blocks

In this section and the preceding section, we have discussed the structure of mathematical proofs and some techniques for solving problems. In the process we have become aware of the general form of the arguments used to prove theorems and the approaches that can be used to devise a proof of a theorem or a solution to a problem. Nonetheless, all of us, no matter how proficient we are at employing problem-solving strategies or knowing proof methods, will occasionally become *stuck* on a problem. We meet what appears to be an insurmountable roadblock and we have no idea how to proceed. What do we do in these instances?

Perhaps the most useful general advice that can be offered is: STAY ACTIVE. Rather than waiting quietly for inspiration to strike, engage in steps that can help bring about a solution. Specifically, how does this advice translate into action? We offer a condensed list of suggestions, many of which have already been mentioned in this section.

1. Read the problem or theorem several times. Rephrase the problem in alternative ways; for example, use quantifiers and connectives; use different but equivalent notation; or use different but equivalent definitions of the terms involved.
2. Play with the problem by drawing pictures (or forming other representations), by checking out examples, extreme cases, and other special cases, and by generalizing in various ways.
3. As a result of this active investigation, try to develop a problem-solving or proof strategy.
4. Execute the chosen strategy and verify the result.

Excerpts from Galovich, S. (1989). *Introduction to Mathematical Structures*. San Diego, CA. Harcourt Brace Jovanovich.

The moral is simple: Keep working, keep trying new things.

While you are actively working on a problem, it is wise to monitor your own activities. For instance, suppose you find an approach that seems promising. But suppose that after trying this approach for a while, you make no further progress. Then you should be able to recognize your situation and be willing to drop your current approach in favor of some other. As another example, suppose you think of two possible ways of solving your problem. Before embarking on either, try to evaluate the relative advantages of the two methods. Perhaps one method will require significantly less work than the other and hence is preferable for this reason. As these remarks indicate, a diligent problem solver should be aware of more than the proof and problem-solving methods discussed in this section and the previous section. Successful problem solving also requires an awareness of one's actions, the flexibility to change actions, and the judgment to choose appropriate methods.

One simple way to monitor and control your actions while working on a problem is to ask yourself repeatedly the following three questions:

1. What am I trying to do?
2. What am I doing?
3. Will what I am doing help me do what I want to do?

If you can answer questions 1 and 2 convincingly and can answer number 3 with a clear Yes, then proceed. Otherwise, stop and take stock of your situation.

A convincing answer to question 1 occurs only when the problem solver has a clear understanding of the problem. A convincing answer to question 2 occurs when the problem solver has a clear understanding of his or her own actions. An affirmative answer to question 3 results only when the actions of the problem solver are relevant to the goals of the problem.

If you cannot give an appropriate answer to any one of these questions, then concentrate on that question. Go back to the Polya–Schoenfeld scheme or to the advice given previously in this subsection. By actively and carefully searching for clues and insights, you greatly increase your ability to devise a successful solution strategy.