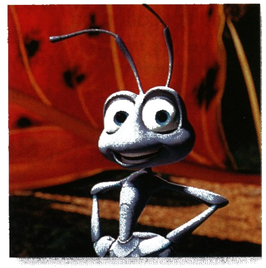
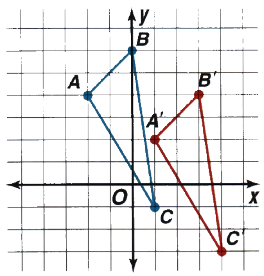
COMPUTER ANIMATION In 1995, animation took a giant step forward with the release of the first major motion picture to be created entirely on computers. Animators use computer software to create three-dimensional computer models of characters, props, and sets. These computer models describe the shape of the object as well as the motion controls that the animators use to create movement and expressions. The animation models are actually very large matrices.



Even though large matrices are used for computer animation, you can use a simple matrix to describe many of the motions called **transformations** that you learned about in geometry. Some of the transformations we will examine in this lesson are **translations** (slides), **reflections** (flips), **rotations** (turns), and **dilations** (enlargements or reductions).

A $2 \times n$ matrix can be used to express the vertices of an n-gon with the first row of elements representing the x-coordinates and the second row the y-coordinates of the vertices.



Triangle ABC can be represented by the following **vertex matrix.**

$$\begin{array}{r c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{r} x\text{-coordinate} \\ y\text{-coordinate} \end{array} & \begin{bmatrix} -2 & 0 & 1 \\ 4 & 6 & -1 \end{bmatrix} \end{array}$$

Triangle $A'B'C'$ is congruent to and has the same orientation as $\triangle ABC$, but is moved 3 units right and 2 units down from $\triangle ABC$'s location. The coordinates of $\triangle A'B'C'$ can be expressed as the following vertex matrix.

$$\begin{array}{r c} & \begin{array}{ccc} A' & B' & C' \end{array} \\ \begin{array}{r} x\text{-coordinate} \\ y\text{-coordinate} \end{array} & \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & -3 \end{bmatrix} \end{array}$$

Compare the two matrices. If you add $\begin{bmatrix} 3 & 3 & 3 \\ -2 & -2 & -2 \end{bmatrix}$ to the first matrix, you get the second matrix. Each 3 represents moving 3 units right for each x-coordinate. Likewise, each -2 represents moving 2 units down for each y-coordinate. This type of matrix is called a **translation matrix.** In this transformation, $\triangle ABC$ is the **pre-image,** and $\triangle A'B'C'$ is the **image** after the translation.

Excerpts from Holliday, B. Cuevas, G. McClure, M. Carter, J. Marks, D. (2007). *Advanced Mathematical Concepts*. New York, NY. McGraw Hill Glencoe.

1 **Suppose quadrilateral $ABCD$ with vertices $A(-1, 1)$, $B(4, 0)$, $C(4, -5)$, and $D(-1, -3)$ is translated 2 units left and 4 units up.**

a. Represent the vertices of the quadrilateral as a matrix.

b. Write the translation matrix.

c. Use the translation matrix to find the vertices of $A'B'C'D'$, the translated image of the quadrilateral.

d. Graph quadrilateral $ABCD$ and its image.

a. The matrix representing the coordinates of the vertices of quadrilateral $ABCD$ will be a 2×4 matrix.

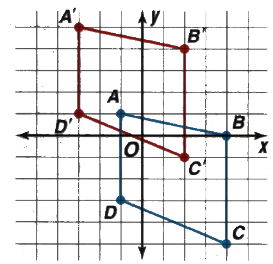
$$\begin{array}{r c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{r} x\text{-coordinate} \\ y\text{-coordinate} \end{array} & \begin{bmatrix} -1 & 4 & 4 & -1 \\ 1 & 0 & -5 & -3 \end{bmatrix} \end{array}$$

b. The translation matrix is $\begin{bmatrix} -2 & -2 & -2 & -2 \\ 4 & 4 & 4 & 4 \end{bmatrix}$.

c. Add the two matrices.

$$\begin{bmatrix} -1 & 4 & 4 & -1 \\ 1 & 0 & -5 & -3 \end{bmatrix} + \begin{bmatrix} -2 & -2 & -2 & -2 \\ 4 & 4 & 4 & 4 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} A' & B' & C' & D' \end{array} \\ \begin{bmatrix} -3 & 2 & 2 & -3 \\ 5 & 4 & -1 & 1 \end{bmatrix} \end{array}$$

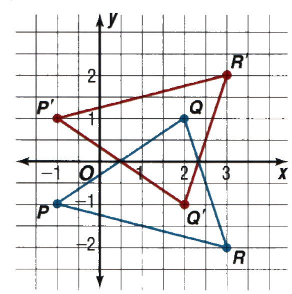
d. Graph the points represented by the resulting matrix.



There are three lines over which figures are commonly reflected.

- the x-axis
- the y-axis, and
- the line $y = x$

In the figure at the right, $\triangle P'Q'R'$ is a reflection of $\triangle PQR$ over the x-axis. There is a 2×2 **reflection matrix** that, when multiplied by the vertex matrix of $\triangle PQR$, will yield the vertex matrix of $\triangle P'Q'R'$.



Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ represent the unknown square matrix.

Thus, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 & 3 \\ -1 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix}$, or

$$\begin{bmatrix} -a-b & 2a+b & 3a-2b \\ -c-d & 2c+d & 3c-2d \end{bmatrix} = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix}.$$

Since corresponding elements of equal matrices are equal, we can write equations to find the values of the variables. These equations form two systems.

$$\begin{aligned} -a - b &= -1 \\ 2a + b &= 2 \\ 3a - 2b &= 3 \end{aligned} \qquad \begin{aligned} -c - d &= 1 \\ 2c + d &= -1 \\ 3c - 2d &= 2 \end{aligned}$$

When you solve each system of equations, you will find that $a = 1$, $b = 0$, $c = 0$, and $d = -1$. Thus, the matrix that results in a reflection over the x-axis is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. This matrix will work for any reflection over the x-axis.

The matrices for a reflection over the y-axis or the line $y = x$ can be found in a similar manner. These are summarized below.

Reflection Matrices

For a reflection over the:	Symbolized by:	Multiply the vertex matrix by:
x-axis	$R_{x\text{-axis}}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
y-axis	$R_{y\text{-axis}}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
line $y = x$	$R_{y = x}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

2 ANIMATION **To create an image that appears to be reflected in a mirror, an animator will use a matrix to reflect an image over the y-axis. Use a reflection matrix to find the coordinates of the vertices of a star reflected in a mirror (the y-axis) if the coordinates of the points connected to create the star are $(-2, 4)$, $(-3.5, 4)$, $(-4, 5)$, $(-4.5, 4)$, $(-6, 4)$, $(-5, 3)$, $(-5, 1)$, $(-4, 2)$, $(-3, 1)$, and $(-3, 3)$.**

First write the vertex matrix for the points used to define the star.

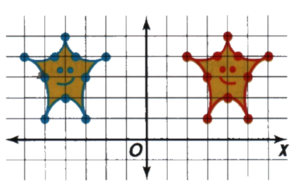
$$\begin{bmatrix} -2 & -3.5 & -4 & -4.5 & -6 & -5 & -5 & -4 & -3 & -3 \\ 4 & 4 & 5 & 4 & 4 & 3 & 1 & 2 & 1 & 3 \end{bmatrix}$$

Multiply by the y-axis reflection matrix.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & -3.5 & -4 & -4.5 & -6 & -5 & -5 & -4 & -3 & -3 \\ 4 & 4 & 5 & 4 & 4 & 3 & 1 & 2 & 1 & 3 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & 3.5 & 4 & 4.5 & 6 & 5 & 5 & 4 & 3 & 3 \\ 4 & 4 & 5 & 4 & 4 & 3 & 1 & 2 & 1 & 3 \end{bmatrix}$$

The vertices used to define the reflection are (2, 4), (3.5, 4), (4, 5), (4.5, 4), (6, 4), (5, 3), (5, 1), (4, 2), (3, 1), and (3, 3).



You may remember from geometry that a rotation of a figure on a coordinate plane can be achieved by a combination of reflections. For example, a 90° counterclockwise rotation can be found by first reflecting the image over the x-axis and then reflecting the reflected image over the line $y = x$. The **rotation matrix**, Rot_{90}, can be found by a composition of reflections. Since reflection matrices are applied using multiplication, the composition of two reflection matrices is a product. Remember that $[f \circ g](x)$ means that you find $g(x)$ first and then evaluate the result for $f(x)$. So, to define Rot_{90}, we use

$$Rot_{90} = R_{y = x} \circ R_{x\text{-axis}} \text{ or } Rot_{90} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Similarly, a rotation of 180° would be rotations of 90° twice or $Rot_{90} \circ Rot_{90}$. A rotation of 270° is a composite of Rot_{180} and Rot_{90}. The results of these composites are shown below.

Rotation Matrices

For a counterclockwise rotation about the origin of:	Symbolized by:	Multiply the vertex matrix by:
90°	Rot_{90}	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
180°	Rot_{180}	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
270°	Rot_{270}	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

3 ANIMATION **Suppose a figure is animated to spin around a certain point. Numerous rotation images would be necessary to make a smooth movement image. If the image has key points at (1, 1), (−1, 4), (−2, 4), and (−2, 3) and the rotation is about the origin, find the location of these points at the 90°, 180°, and 270° counterclockwise rotations.**

First write the vertex matrix. Then multiply it by each rotation matrix.

The vertex matrix is $\begin{bmatrix} 1 & -1 & -2 & -2 \\ 1 & 4 & 4 & 3 \end{bmatrix}$.

$$Rot_{90} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -2 & -2 \\ 1 & 4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -4 & -4 & -3 \\ 1 & -1 & -2 & -2 \end{bmatrix}$$

$$Rot_{180} \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -2 & -2 \\ 1 & 4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 & 2 \\ -1 & -4 & -4 & -3 \end{bmatrix}$$

$$Rot_{180} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -2 & -2 \\ 1 & 4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 4 & 3 \\ -1 & 1 & 2 & 2 \end{bmatrix}$$