Rational Functions

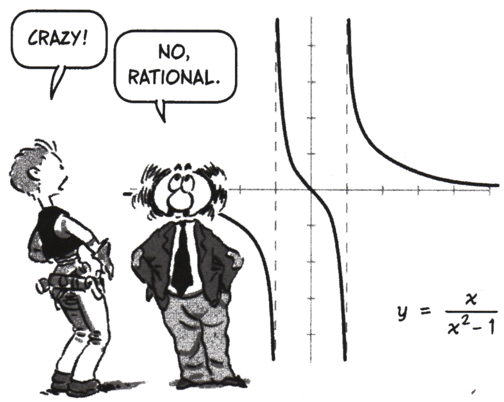
THESE ARE FUNCTIONS GIVEN BY **RATIOS** OF POLYNOMIALS

$$R(x) = \frac{P(x)}{Q(x)}$$

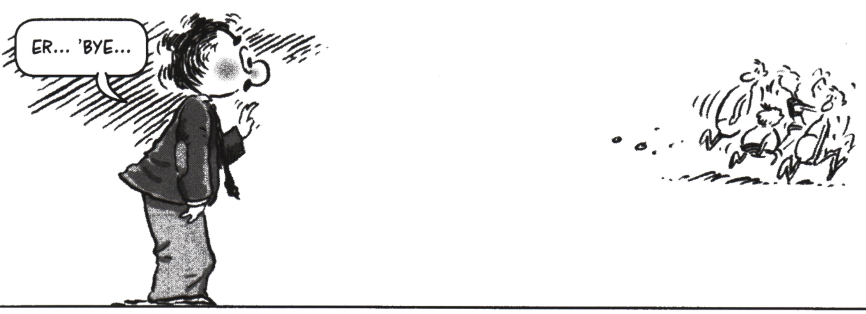
THEY ARE DEFINED WHEREVER $Q(x) \neq 0$. FOR EXAMPLE,

$$R(x) = \frac{3x^2 + 9x + 1}{x^3 + 16}, \quad x \neq \sqrt[3]{-16}$$

$$T(x) = \frac{x}{x^2 - 1}, \quad x \neq \pm 1$$



WE HAVE THREE THINGS TO SAY ABOUT RATIONAL FUNCTIONS. FIRST IS THAT YOU CAN SKIP THIS SECTION AND HEAD FOR PAGE 29 IF YOU WANT TO...



SECOND, WE CAN ASSUME THAT P HAS LOWER DEGREE THAN Q. IF IT DOESN'T, YOU CAN DO **LONG DIVISION OF POLYNOMIALS*** TO MAKE P/Q LOOK LIKE

$$P_1(x) + \frac{R(x)}{Q(x)}$$

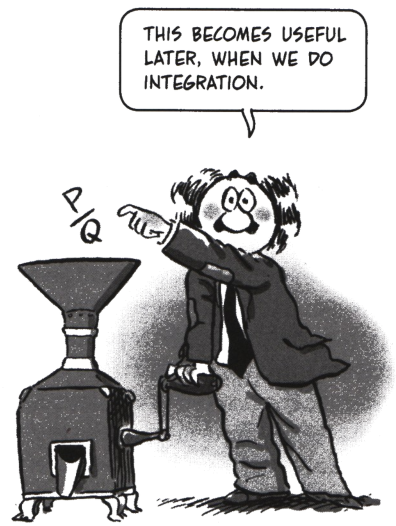
WHERE P_1 IS A POLYNOMIAL, AND R, THE REMAINDER, IS A POLYNOMIAL WITH DEGREE LOWER THAN THAT OF Q.

*IF YOU'VE NEVER DONE LONG DIVISION OF POLYNOMIALS, IT'S JUST LIKE LONG DIVISION OF NUMBERS, ONLY EASIER. LOOK IT UP SOMEWHERE; YOU'LL LIKE IT!

THIRD, ANY RATIONAL FUNCTION CAN BE WRITTEN AS A **SUM** OF SIMPLER "PARTIAL FRACTIONS" OF THESE TWO KINDS:

$$\frac{a}{(x+p)^n} \quad \text{OR} \quad \frac{bx+c}{(x^2+qx+r)^m},$$

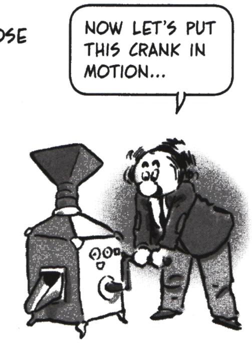
WHERE a, b, c, p, q, AND r ARE CONSTANTS, AND n AND m ARE POSITIVE INTEGERS. IN OTHER WORDS, THE DENOMINATORS ARE POWERS OF FIRST- OR SECOND-DEGREE POLYNOMIALS.



FINDING THESE CONSTANTS CAN BE MESSY IN PRACTICE—FOR STARTERS, YOU HAVE TO FACTOR $Q(x)$—BUT HERE ARE TWO EXAMPLES TO SHOW HOW IT WORKS.

Example: SUPPOSE

$$F(x) = \frac{x}{(x-1)^2}$$



FIRST WRITE IT AS

$$\left(\frac{x}{x-1}\right)\left(\frac{1}{x-1}\right)$$

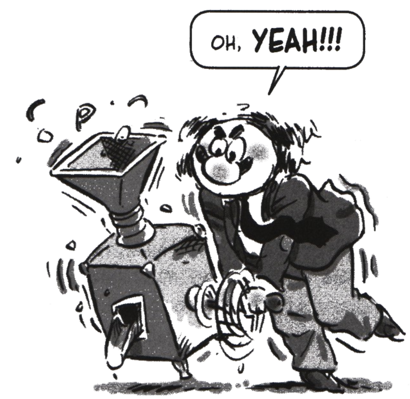
THE FIRST FACTOR CAN BE REDUCED BY LONG DIVISION:

$$\left(\frac{x}{x-1}\right) = \frac{1}{x-1} + 1$$

PLUGGING THAT IN AND EXPANDING GIVES

$$\left(\frac{1}{x-1} + 1\right)\left(\frac{1}{x-1}\right) = \frac{1}{(x-1)^2} + \frac{1}{x-1}$$

JUST AS PROMISED—NOTHING BUT CONSTANTS IN THE NUMERATORS OF FRACTIONS WITH DENOMINATORS OF THE FORM $(x+p)^n$.



Excerpts from Gonick, Larry (2012). *The Cartoon Guide to Calculus*. New York, NY. Harper Collins.

Example:

$$R(x) = \frac{-2x^2 + 7x - 3}{x^3 + 1}$$

THE FIRST STEP IS ALWAYS TO FACTOR THE DENOMINATOR. RECALL FROM ALGEBRA:

$$x^3 + 1 = (x + 1)(x^2 - x + 1).$$

NOW, ASSUME THERE IS AN ANSWER.

IT WOULD LOOK LIKE THIS:

$$\frac{-2x^2+7x-3}{x^3+1} = \frac{Ax+B}{(x^2-x+1)} + \frac{C}{x+1}$$

WE WISH TO SOLVE FOR A, B, AND C. COMBINING THE FRACTIONS ON THE RIGHT PRODUCES THIS NUMERATOR:

$$(A + C)x^2 + (A + B - C)x + (B + C)$$

THIS BEING THE SAME AS THE NUMERATOR OF THE ORIGINAL FRACTION, WE MUST HAVE

$$A + C = -2$$
$$A + B - C = 7$$
$$B + C = -3$$

THESE ARE THREE EQUATIONS IN THREE UNKNOWNS. WE DO SOME ALGEBRA AND FIND...

$A = 2$, $B = 1$, AND $C = -4$, SO:

$$R(x) = \frac{2x + 1}{x^2 - x + 1} + \frac{-4}{x + 1}$$

YOU CAN CHECK THE ANSWER BY ADDING TOGETHER THESE FRACTIONS, WHICH SHOULD COMBINE TO GIVE THE ORIGINAL FUNCTION.

AND NOW FOR SOMETHING YOU WON'T WANT TO MISS... THIS NEXT FUNCTION WILL REALLY GROW ON YOU...

Excerpts from Gonick, Larry (2012). *The Cartoon Guide to Calculus*. New York, NY. Harper Collins.

Exponential Functions

EXPONENTIAL FUNCTIONS ARE GIVEN BY FORMULAS LIKE THIS:

$$f(x) = a^x$$

HERE THE "BASE" a IS FIXED, AND THE EXPONENT x VARIES. BY CONVENTION, WE ASSUME $a > 1$. THESE FUNCTIONS DESCRIBE CERTAIN KINDS OF GROWTH (POPULATION INCREASE, FOR EXAMPLE).

AMONG ALL POSSIBLE BASES a, MATHEMATICIANS SINGLE OUT ONE AS ESPECIALLY "NATURAL." THIS NUMBER, KNOWN AS e, HAS A DECIMAL EXPANSION THAT BEGINS LIKE THIS:

2.71828182845904523536028747135266249775724709369995957496696762772407663
03535475945713821785251664274274663919320030599218174135966290435729003342
95260595630738132328627943490763233829880753195251019011573834187930702154
08914993488416750924476146066808226480016847741185374234544243710753907774
49920695517027618386062613313845830007520449338265602976067371132007093287
09127443747047230696977209310141692836819025515108657463772111252389784425
05695369677078544996996794686445490598793163688923009879312773617821542499
92295763514822082698951936680331825288693984964651058209392398294887933203
62509443117301238197068416140397019837679320683282376464804295311802328782
50981945581530175671736133206981125099618188159304169035159888851934580727
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24827041978623209002160990235304369941849146314093431738143640546253152096
18369088870701676839642437814059271456354906130 37505101
1574770417189861068739696552126715468895703503540212340 10681701
210056278802351930332247450158539047304199577770935036604169973297250886
876966403555707162268447162560798826517871341951246652010305921236677194325
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3648961342251644507818244235294863637214174023889344124 7 96357437
0263755294448337998016125492278509257782562092622648326 62779333
865664816277251640191059004916449982893150566047258027 7863186415519
56532442586982946959308019152987211725563475463964479 1 01459040905
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