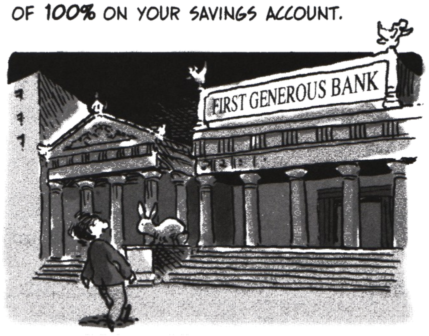
WE CAN SEE WHY e IS NATURAL BY THINKING ABOUT **COMPOUND INTEREST.** IMAGINE A GENEROUS BANK (!) IS PAYING ANNUAL INTEREST OF **100%** ON YOUR SAVINGS ACCOUNT.

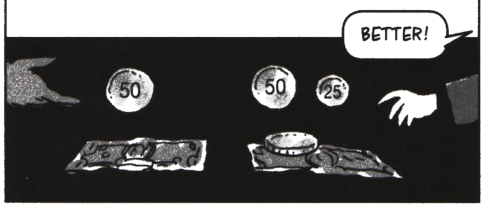


IF YOU START WITH \$1, AT THE END OF THE YEAR YOUR ACCOUNT WOULD HAVE DOUBLED TO \$2. PRETTY GOOD!

$$\$1 + 100\% \cdot (\$1) = \$2$$

BUT NOT GOOD ENOUGH, YOU COMPLAIN: YOU WANT YOUR INTEREST COMPOUNDED **MORE OFTEN.** YOU ASK THE BANK TO ADD ON 50% EVERY SIX MONTHS (100% PER YEAR TIMES HALF A YEAR), FOR THIS YEAR-END DOLLAR TOTAL:

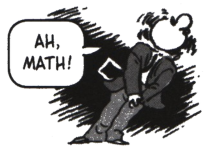
$$(1 + \tfrac{1}{2}) + \tfrac{1}{2}(1 + \tfrac{1}{2}) = 2.25$$



NOW YOU DO A LITTLE ARITHMETIC: YOU NOTICE THAT

$$(1 + \tfrac{1}{2}) + \tfrac{1}{2}(1 + \tfrac{1}{2}) = (1 + \tfrac{1}{2})^2$$

AND THE NEXT TIME INTEREST IS ADDED, YOUR DOLLAR TOTAL WILL BE $(1 + \frac{1}{2})^3$, NEXT TIME $(1 + \frac{1}{2})^4$, NEXT TIME $(1 + \frac{1}{2})^5$...



SIMILARLY, IF YOU COMPOUND AT 100% **THREE** TIMES A YEAR, YOUR TOTAL AFTER ONE YEAR (THREE PAYMENTS) IS

$$\$(1 + \tfrac{1}{3})^3$$

IF COMPOUNDED n TIMES PER YEAR, YOUR YEAR-END TOTAL WOULD BE

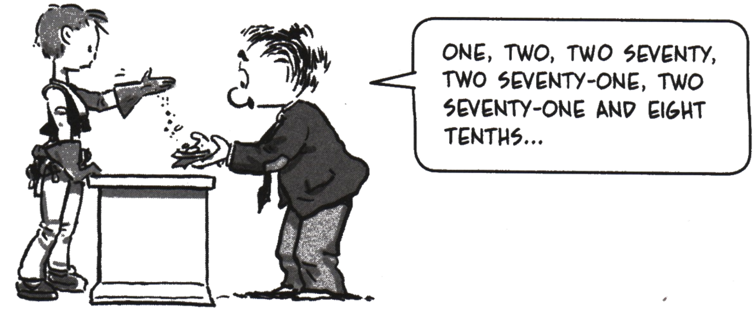
$$\$(1 + \tfrac{1}{n})^n$$

AND YOU DECIDE TO FIND OUT JUST HOW MUCH MONEY THIS WOULD BE! USING YOUR CALCULATOR, YOU FIND:

PAYMENTS PER YEAR		TOTAL AFTER 1 YEAR
1	$(1+1)^1$	= \$2
2	$(1+\frac{1}{2})^2$	= \$2.25
3	$(1+\frac{1}{3})^3$	≈ \$2.37
4	$(1+\frac{1}{4})^4$	≈ \$2.44
5	$(1+\frac{1}{5})^5$	≈ \$2.49
	...	
100	$(1+\frac{1}{100})^{100}$	≈ \$2.705...
1000	$(1+\frac{1}{1000})^{1000}$	≈ \$2.718...
	...	

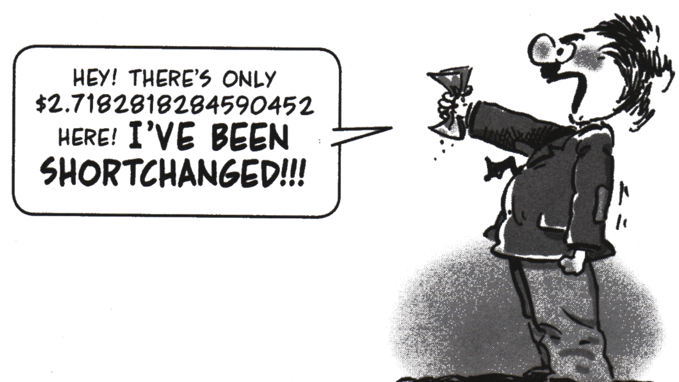
Excerpts from Gonick, Larry (2012). *The Cartoon Guide to Calculus*. New York, NY. Harper Collins.

IF n IS VERY, VERY LARGE, YOU CAN THINK OF YOUR MONEY AS BEING COMPOUNDED **CONTINUOUSLY, ALL THE TIME.** IN THAT CASE, YOUR TOTAL BALANCE AT THE END OF ONE YEAR WOULD BE **EXACTLY e DOLLARS.**



THE NUMBER e IS NATURAL BECAUSE CONTINUOUS COMPOUNDING IS NATURAL: IT DOESN'T DEPEND ON ANY PARTICULAR UNIT OF TIME.

THIS ALSO SHOWS THAT e IS THE MOST YOU CAN POSSIBLY MAKE IN A YEAR FROM ONE DOLLAR AT 100% INTEREST!



WE CAN USE THE FORMULA $(1+\frac{1}{n})^n$ TO CALCULATE e. ALGEBRA TELLS US WE CAN EXPAND THAT BINOMIAL AS

$$1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2}\cdot\frac{1}{n^2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cdot\frac{1}{n^3} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{1}{n^4} + \cdots + \frac{1}{n^n}$$

WHEN n IS VERY LARGE, THE FRACTIONS $(n-1)/n$, $(n-2)/n$, ETC. ARE VERY NEARLY EQUAL TO 1, SO THE EARLY TERMS ARE VERY NEARLY

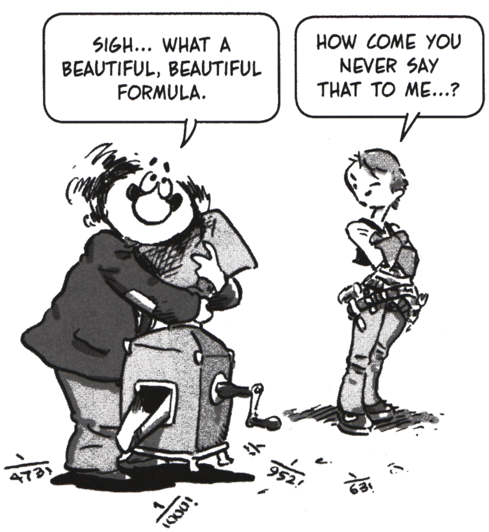
$$1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

WHERE, IF m IS ANY INTEGER, $m!$ MEANS THE PRODUCT $1\cdot 2\cdot 3\cdot \ldots \cdot m$.

NOW IF WE IMAGINE n GROWING "TO ∞," WE CAN CONCLUDE THAT e IS GIVEN BY A SUM WITH AN **INFINITE** NUMBER OF TERMS:

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots + \frac{1}{n!} + \cdots$$

AND SO, IN FACT, IT IS.



BECAUSE OF THIS NUMBER'S SPECIAL, NATURAL STATUS, FROM NOW ON WE WILL REFER TO THE FUNCTION **exp**, DEFINED BY

$$\exp(x) = e^x$$

AS **THE** EXPONENTIAL FUNCTION. e^x IS THE SUM YOU WOULD HAVE AFTER x YEARS IF ONE DOLLAR WERE COMPOUNDED CONTINUOUSLY AT 100% PER YEAR.

Excerpts from Gonick, Larry (2012). *The Cartoon Guide to Calculus*. New York, NY. Harper Collins.

EXPONENTIAL FUNCTIONS GROW RAPIDLY WITH x. $f(x) = 2^x$, FOR EXAMPLE, DOUBLES EVERY TIME x INCREASES BY 1:

$$f(x+1) = 2^{x+1} = 2^x 2^1 = 2(2^x) = 2f(x)$$

e^x GROWS EVEN FASTER, AS YOU CAN EASILY CALCULATE. A POWER FUNCTION LIKE $g(x) = x^2$, BY COMPARISON, FALLS FAR BEHIND.

x	e^x	x^2
0	1.0	0
1	2.7183...	1
2	7.389...	4
3	20.085...	9
4	54.60...	16
5	148.41...	25
6	403.43...	36
7	1096.63...	49
8	2980.94...	64

IF a IS THE NUMBER WITH $e^a = 2$ ($a \approx 0.693$, AS YOU CAN CHECK ON YOUR CALCULATOR), THEN e^x DOUBLES WHENEVER x INCREASES BY a:

$$e^{(x+a)} = e^x e^a = 2e^x$$

AND IN PARTICULAR,

$$e^{na} = (e^a)^n = 2^n$$