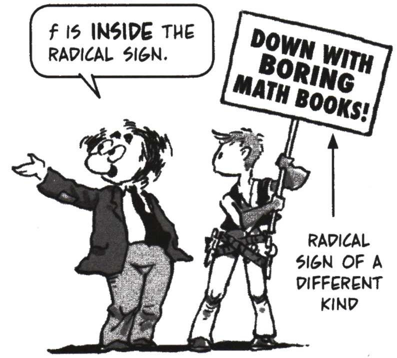
ANOTHER BASIC IDEA:

Composing Functions

SOMETIMES ONE FUNCTION IS "PLUGGED INTO" ANOTHER FUNCTION. FOR EXAMPLE, ON P. 15,

$$h(x) = \sqrt{x^2 - 1}$$

IS THE RESULT OF PLUGGING THE VALUE OF $f(x) = x^2 - 1$ INTO THE SQUARE ROOT FUNCTION $g(u) = \sqrt{u}$. FIRST WE EVALUATE $x^2 - 1$ AND THEN TAKE THE SQUARE ROOT. f IS CALLED THE **INSIDE** FUNCTION, AND g IS THE **OUTSIDE** FUNCTION.



Example 1:

$F(x) = \tan^2 x + \tan x + 1$

FIRST FIND $\tan x$, THEN PLUG IT INTO $g(y) = y^2 + y + 1$. THE INSIDE FUNCTION IS $f(x) = \tan x$ AND THE OUTSIDE FUNCTION IS g. WE WRITE

$$F(x) = g(f(x))$$

Example 2:

$G(x) = e^{x^2}$

INSIDE FUNCTION:

$u(x) = x^2$

OUTSIDE FUNCTION:

$v(t) = e^t$

$$G(x) = v(u(x))$$

Example 3:

$H(x) = \tan(x^2 + x + 1)$

INSIDE FUNCTION:

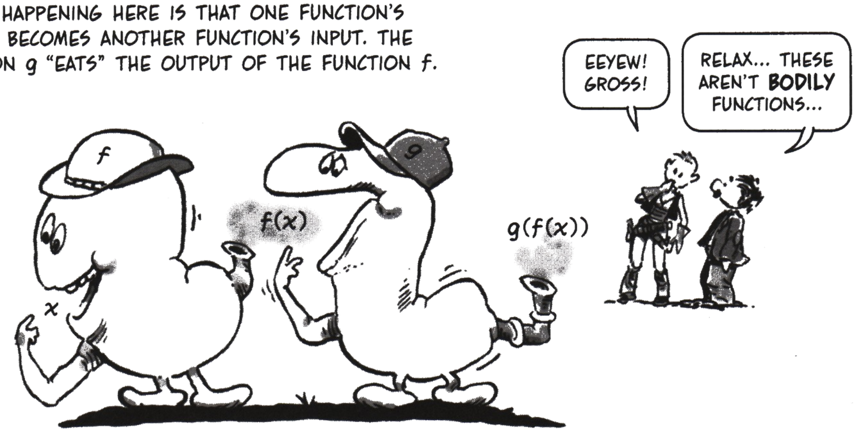
$g(x) = x^2 + x + 1$

OUTSIDE FUNCTION:

$f(\theta) = \tan\theta$

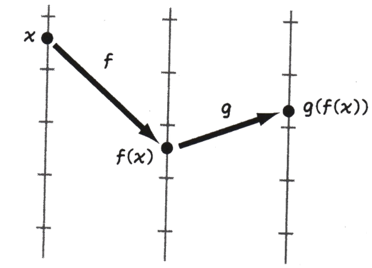
$$H(x) = f(g(x))$$

WHAT'S HAPPENING HERE IS THAT ONE FUNCTION'S OUTPUT BECOMES ANOTHER FUNCTION'S INPUT. THE FUNCTION g "EATS" THE OUTPUT OF THE FUNCTION f.



Excerpts from Gonick, Larry (2012). *The Cartoon Guide to Calculus*. New York, NY. Harper Collins.

IN EFFECT, THE ARROW OF f IS FOLLOWED BY THE ARROW OF g:

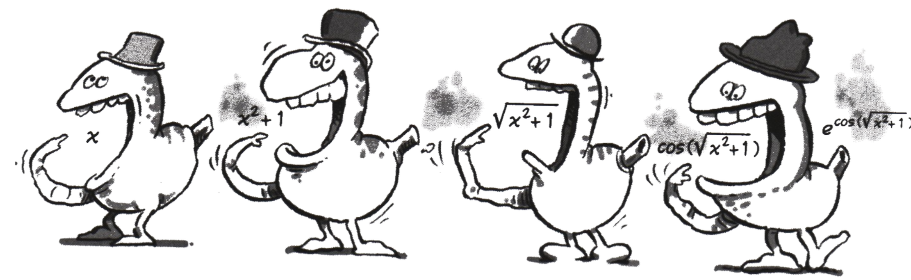


$$h : x \rightarrow f(x) \rightarrow g(f(x))$$

WE CALL THE FUNCTION h THE **COMPOSITION** OF g AND f, SOMETIMES WRITTEN $g \circ f$. NOTE THAT THE **INSIDE FUNCTION IS EVALUATED FIRST.** ITS ARROW IS ON THE **LEFT.** ALSO NOTE THAT **THE ORDER MATTERS.** IN GENERAL, $g \circ f \neq f \circ g$. IN EXAMPLES 1 AND 3 ON THE PREVIOUS PAGE, FOR INSTANCE,

$$f(g(x)) = \tan(x^2 + x + 1)$$
$$\neq \tan^2 x + \tan x + 1 = g(f(x))$$

YOU CAN EVEN HAVE A CHAIN COMPOSED OF MANY FUNCTIONS. WHY NOT!?



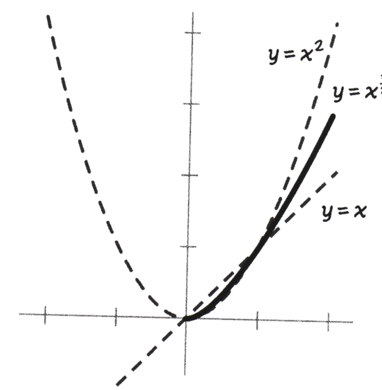
COMPOSITION LEADS STRAIGHT TO

Fractional Powers

BY COMPOSING $f(x) = x^{\frac{1}{n}}$ WITH $g(y) = y^m$, WE CAN DEFINE FRACTIONAL POWERS OF x:

$$h(x) = x^{\frac{m}{n}} = (x^{\frac{1}{n}})^m = (x^m)^{\frac{1}{n}}.$$

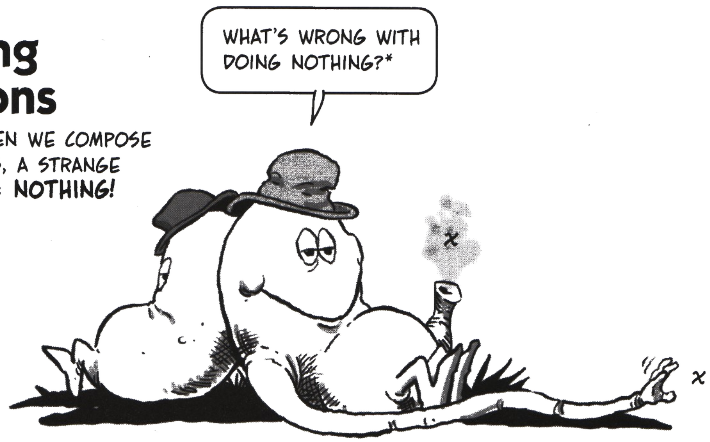
FIRST TAKE THE nTH ROOT AND THEN THE mTH POWER, OR VICE VERSA. (HERE THE ORDER OF COMPOSITION DOESN'T MATTER.)



NEXT BIG IDEA:

Inverting Functions

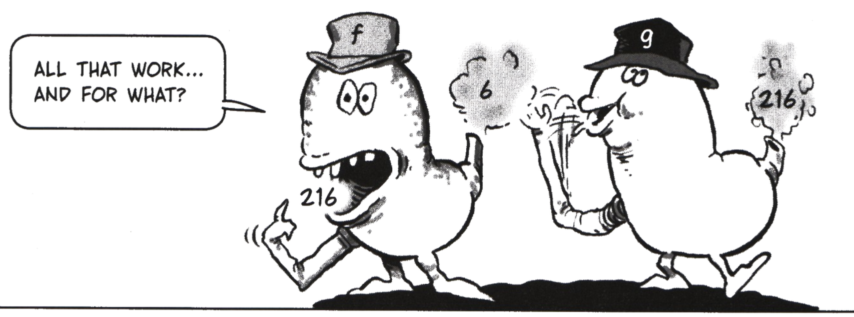
SOMETIMES WHEN WE COMPOSE TWO FUNCTIONS, A STRANGE THING HAPPENS: **NOTHING!**



Example: IF

$f(x) = x^{\frac{1}{3}}$ AND $g(y) = y^3$ THEN $h(x) = g(f(x)) = (x^{\frac{1}{3}})^3 = x$

PLUG x INTO $g \circ f$, AND OUT COMES x AGAIN. h CUBES THE CUBE ROOT, SO IN THE END THIS COMPOSITION DOESN'T DO ANYTHING! g "UNDOES" THE EFFECT OF f.



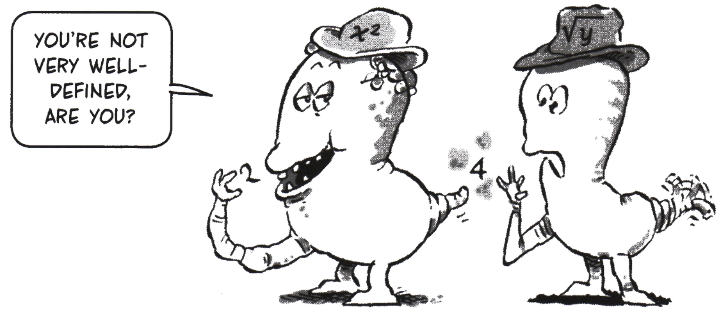
IN WORDS, $g(x)$ IS "THE NUMBER WHOSE CUBE IS x." WE OFTEN WANT TO KNOW THIS KIND OF INFORMATION... SUCH THINGS AS:

THE NUMBER WHOSE SQUARE IS 4 THE NUMBER WHOSE SINE IS $\frac{1}{2}\sqrt{2}$ THE NUMBER WHOSE EXPONENTIAL IS 2	OR, IN SYMBOLS, WHAT NUMBER x, θ, OR t SOLVES THE EQUATIONS:	$x^2 = 4$ $\sin\theta = \frac{1}{2}\sqrt{2}$ $e^t = 2$

*WITH A TIP OF THE HAT TO THE CHINESE PHILOSOPHER ZHUANGZI!

Excerpts from Gonick, Larry (2012). *The Cartoon Guide to Calculus*. New York, NY. Harper Collins.

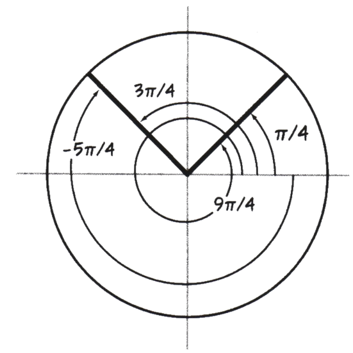
BUT THERE'S A COMPLICATION... IT UNFORTUNATELY MAKES NO SENSE TO ASK FOR "THE" NUMBER WHOSE SQUARE IS 4, BECAUSE THERE ARE TWO OF THEM, 2 AND −2.



THE SINE IS EVEN WORSE. THE ANGLE $\pi/4$ SOLVES THE EQUATION:

$$\sin\theta = \tfrac{1}{2}\sqrt{2}$$

BUT SO DO A LOT OF OTHER ANGLES: $3\pi/4$, $-5\pi/4$, $9\pi/4$, $11\pi/4$, ETC.



$$\sin\left(\frac{\pi}{4} \pm 2\pi n\right) = \tfrac{1}{2}\sqrt{2}, \quad n = 0, 1, 2, 3, \ldots$$

$$\sin\left(\frac{3\pi}{4} \pm 2\pi n\right) = \tfrac{1}{2}\sqrt{2}, \quad n = 0, 1, 2, 3, \ldots$$

IN OTHER WORDS, THESE FUNCTIONS HAVE **MANY ARROWS** LANDING ON THE GIVEN NUMBER. A VALUE OF THE FUNCTION GENERALLY COMES FROM MANY DIFFERENT VALUES OF x.