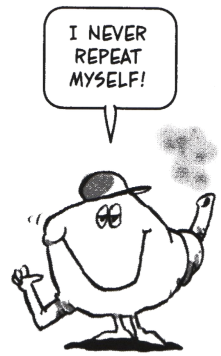
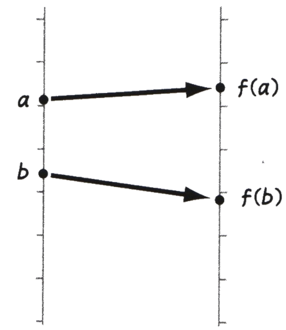
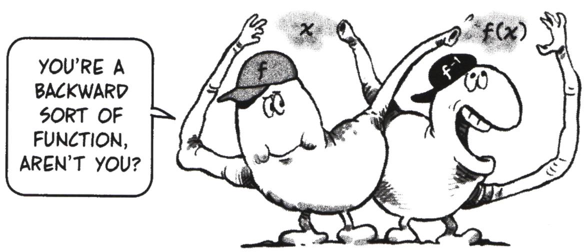
BUT NOT ALL FUNCTIONS ARE LIKE THAT: A FUNCTION IS CALLED **ONE-TO-ONE** IF NO TWO OF ITS ARROWS LAND IN THE SAME PLACE. IN SYMBOLS, IF $a \neq b$, THEN $f(a) \neq f(b)$. EACH VALUE OF f IS THE HEAD OF ONLY **ONE ARROW.**

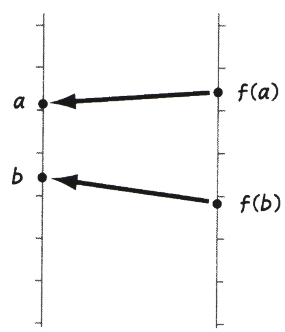




IF f IS ANY ONE-TO-ONE FUNCTION, WE CAN MAKE A NEW FUNCTION, f^{-1}, "***f*-INVERSE,**" THAT UNAMBIGUOUSLY UNDOES THE ACTION OF f BY **REVERSING ITS ARROWS.** THE DOMAIN OF THE INVERSE FUNCTION f^{-1} IS ALL THE VALUES ASSUMED BY f, AND FOR ANY NUMBER $f(x)$ IN ITS DOMAIN, f^{-1} IS DEFINED BY

$$f^{-1}(f(x)) = x$$

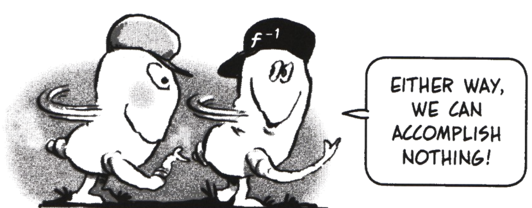


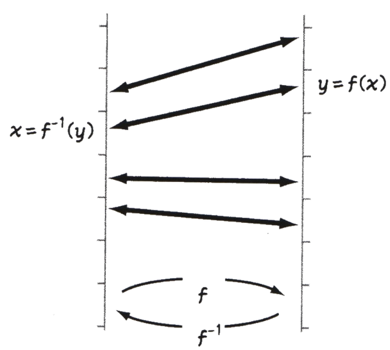


BECAUSE f^{-1} REVERSES THE ARROWS OF f, f OBVIOUSLY REVERSES THE ARROWS OF f^{-1} TOO—IT'S MUTUAL! SO IT FOLLOWS THAT

$$f(f^{-1}(y)) = y$$

THE TWO FUNCTIONS ARE INVERSES OF EACH OTHER! ORDER DOESN'T MATTER.





Excerpts from Gonick, Larry (2012). *The Cartoon Guide to Calculus*. New York, NY. Harper Collins.

WHAT FUNCTIONS ARE ONE-TO-ONE? FOR OUR PURPOSES, IT WILL BE FUNCTIONS THAT ARE

increasing or decreasing.

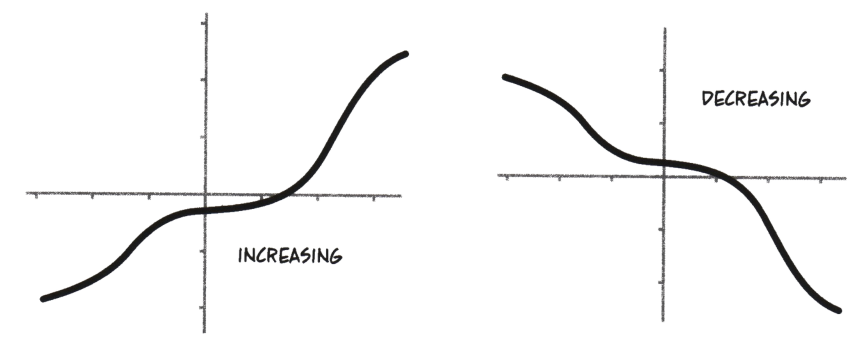
WE DEFINE A FUNCTION TO BE **INCREASING,** OR **STRICTLY INCREASING,** IF THE VALUES $f(x)$ RISE AS x DOES. THAT IS, GIVEN ANY TWO POINTS a AND b IN THE DOMAIN OF f,

IF $a < b$, THEN $f(a) < f(b)$.

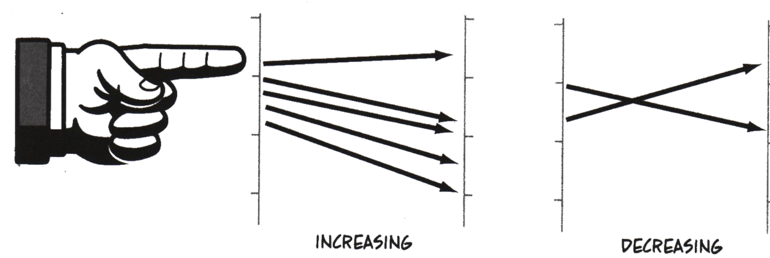
f IS **STRICTLY DECREASING** IF $a < b$ IMPLIES THAT $f(a) > f(b)$.* BECAUSE OF THE INEQUALITY, **EVERY INCREASING FUNCTION IS ONE-TO-ONE,** AND SO IS EVERY DECREASING FUNCTION.

THE VOLUME OF A SPHERE IS AN INCREASING FUNCTION OF RADIUS

AN INCREASING FUNCTION HAS A GRAPH THAT GOES UPHILL AS THE VARIABLE MOVES TO THE RIGHT. A DECREASING FUNCTION GOES DOWNHILL.



IN TERMS OF ARROWS, AN INCREASING FUNCTION'S ARROWS NEVER CROSS, BECAUSE THE VALUES $f(x)$ KEEP GOING UP THE LINE. **ALL** A DECREASING FUNCTION'S ARROWS CROSS EACH OTHER!



*NOTE THAT A FUNCTION f IS INCREASING IF AND ONLY IF $-f$ IS DECREASING.

SINCE AN INCREASING (OR DECREASING) FUNCTION IS ONE-TO-ONE, IT HAS AN INVERSE!

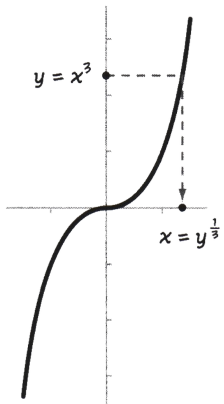
Little Example:

$f(x) = x^3$ IS INCREASING. ITS INVERSE IS

$$f^{-1}(x) = x^{\frac{1}{3}}$$

IN GENERAL, $g(x) = x^n$ IS INCREASING FOR ANY **ODD** INTEGER n, AND THE INVERSE IS

$$g^{-1}(x) = x^{\frac{1}{n}}$$



Big, Important Example: Natural Logarithm, Inverse of the Exponential

THE **EXPONENTIAL FUNCTION** $Exp(x) = e^x$ IS INCREASING.

PROOF: IF $a < b$, THEN

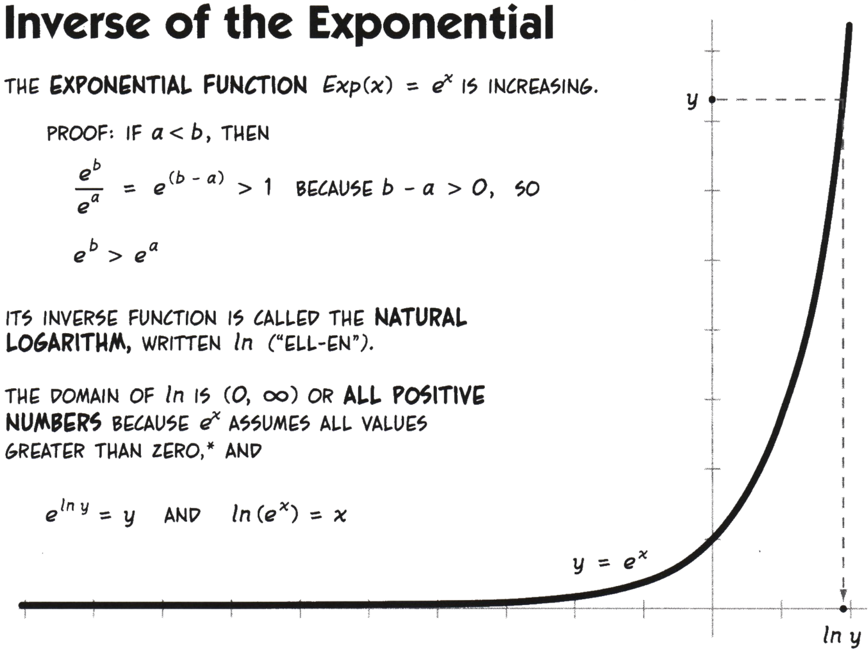
$$\frac{e^b}{e^a} = e^{(b-a)} > 1 \quad \text{BECAUSE } b - a > 0, \text{ SO}$$

$$e^b > e^a$$

ITS INVERSE FUNCTION IS CALLED THE **NATURAL LOGARITHM,** WRITTEN ln ("ELL-EN").

THE DOMAIN OF ln IS $(0, \infty)$ OR **ALL POSITIVE NUMBERS** BECAUSE e^x ASSUMES ALL VALUES GREATER THAN ZERO,* AND

$$e^{\ln y} = y \quad \text{AND} \quad \ln(e^x) = x$$



*SORRY, BUT YOU'RE ASKED TO TAKE THIS ON FAITH IN THIS BOOK.

Excerpts from Gonick, Larry (2012). *The Cartoon Guide to Calculus*. New York, NY. Harper Collins.

EXPONENTS, YOU SHOULD RECALL, BEHAVE THIS WAY:

$$(e^x)(e^y) = e^{x+y} \qquad (e^x)^y = e^{xy}$$

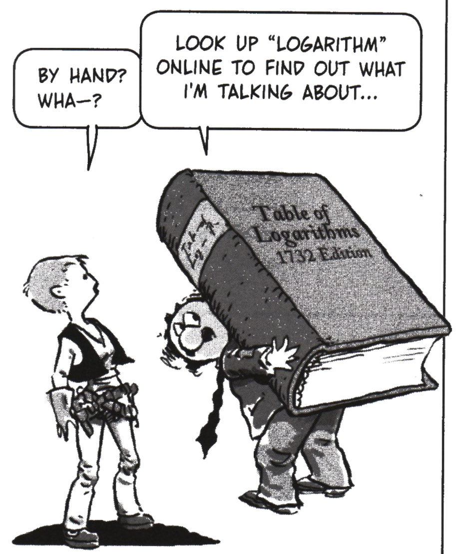
THESE IMPLY THE FAMOUS LOG FORMULAS THAT USED TO BE SO IMPORTANT FOR MANAGING BIG CALCULATIONS BACK IN THE DAY BEFORE MECHANICAL AND ELECTRONIC COMPUTERS, WHEN EVERYTHING WAS DONE BY HAND.

$$\ln(xy) = \ln x + \ln y$$

$$\ln x^p = p \ln x$$

AND IN PARTICULAR, WHEN $p = -1$,

$$\ln \frac{1}{x} = \ln x^{-1} = -\ln x$$



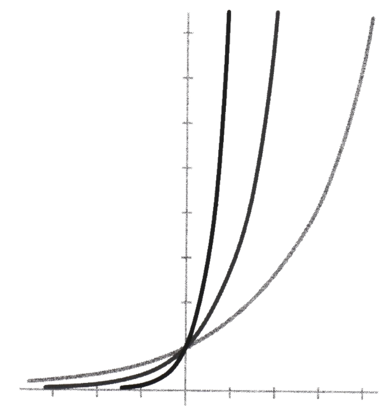
THE LOGARITHM ENABLES US TO EXPRESS OTHER EXPONENTIALS IN TERMS OF "THE" EXPONENTIAL WITH BASE e. TAKE 2^x, FOR EXAMPLE. USING A CALCULATOR, YOU CAN FIND AN APPROXIMATE VALUE FOR $\ln 2$:

$$\ln 2 \approx 0.693...^* \quad \text{FROM WHICH:}$$

$$2^x = (e^{\ln 2})^x = e^{(\ln 2)x} = e^{0.693...x}$$

REPLACE 2 BY **ANY** NUMBER $a > 1$ AND THE EXPONENTIAL $A(x) = a^x$ CAN BE EXPRESSED SIMILARLY:

$$a^x = e^{rx}, \text{ WHERE } r = \ln a.$$



CONCLUSION: **EVERY EXPONENTIAL FUNCTION CAN BE EXPRESSED AS e^{rx} FOR SOME NUMBER r.**

*IT MAKES SENSE THAT $\ln 2$ IS BETWEEN 0 AND 1, BECAUSE 2 IS BETWEEN 1 ($= e^0$) AND e ($= e^1$).