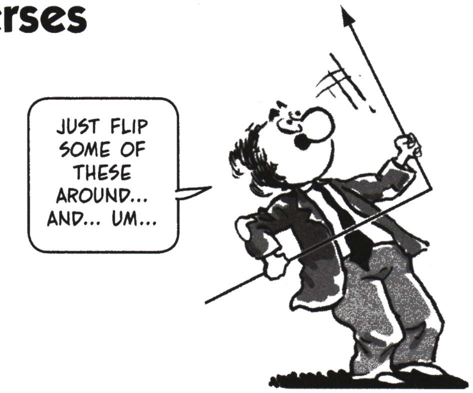
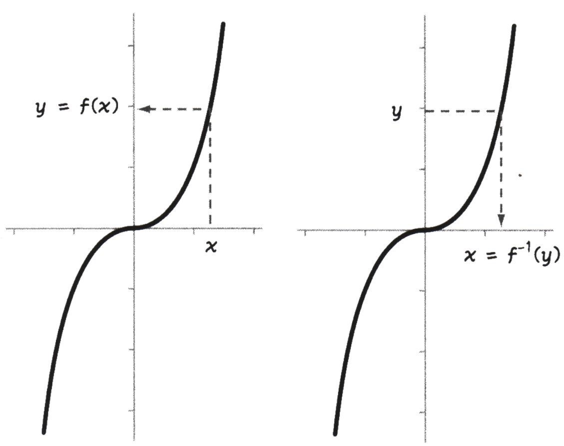
Graphing Inverses

WE'VE SEEN HOW INVERSES LOOK IN TERMS OF ARROWS: f^{-1} SIMPLY TURNS ALL THE f ARROWS AROUND. HOW DOES THIS LOOK ON A GRAPH?



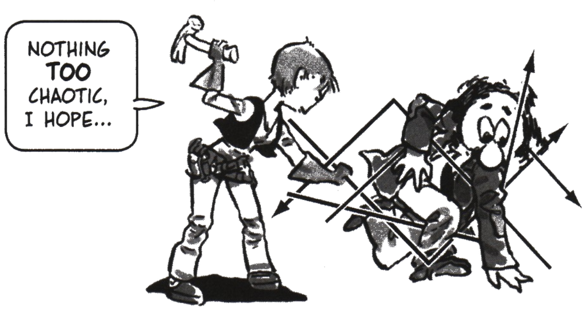
ON THE GRAPH $y = f(x)$, FOLLOW AN ARROW FROM A POINT x TO $f(x) = y$. THE INVERSE FUNCTION f^{-1} REVERSES THAT ARROW, SO $f^{-1}(y) = x$.



THAT IS, IF WE USE THE VERTICAL y-AXIS FOR THE INDEPENDENT VARIABLE, THE GRAPH $x = f^{-1}(y)$ IS **IDENTICAL** TO THE GRAPH $y = f(x)$!

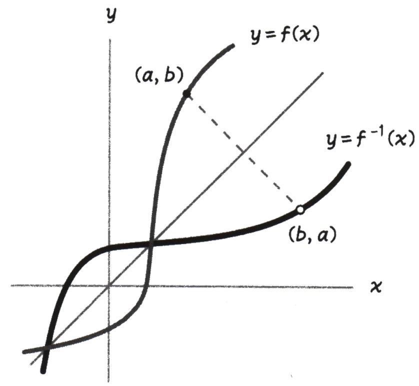
UNFORTUNATELY, WE CUSTOMARILY PUT THE INDEPENDENT VARIABLE ON THE **HORIZONTAL** AXIS, NOT THE VERTICAL AXIS. WE WANT THE GRAPH $y = f^{-1}(x)$, **NOT** $x = f^{-1}(y)$.

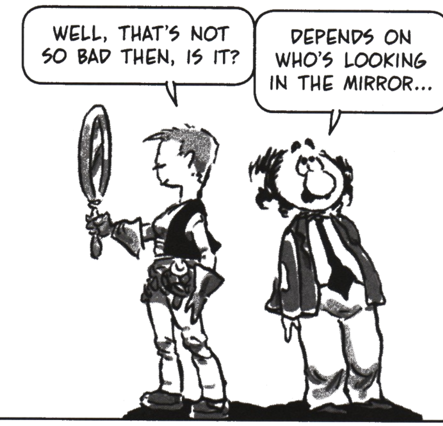
WHAT HAPPENS IF WE EXCHANGE x AND y?

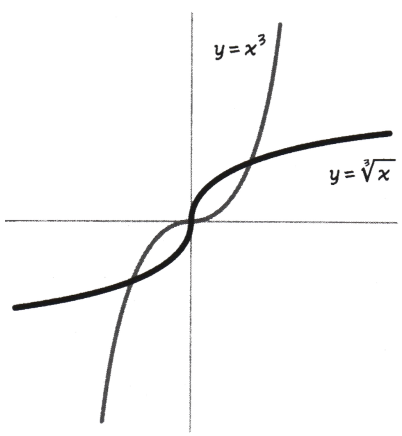


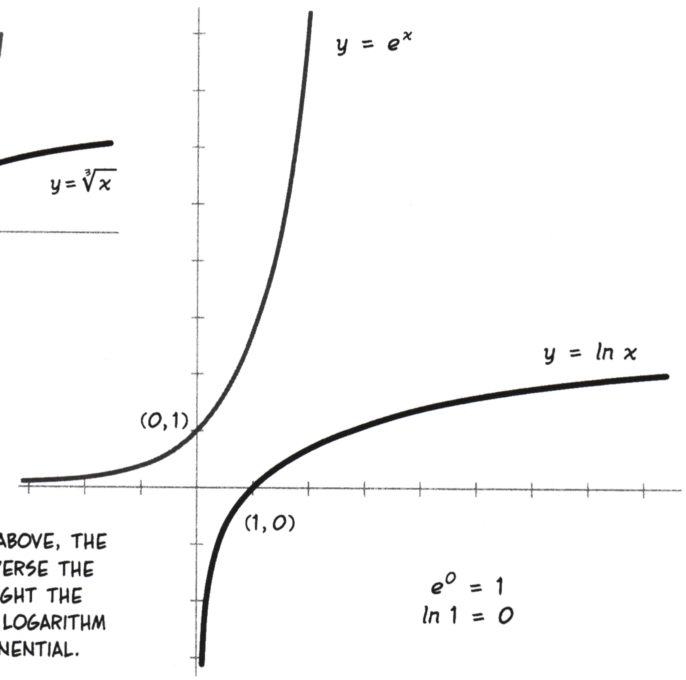
Excerpts from Gonick, Larry (2012). *The Cartoon Guide to Calculus*. New York, NY. Harper Collins.

IF A POINT (a, b) IS ON THE GRAPH $y = f(x)$, THEN (b, a) IS ON THE GRAPH $y = f^{-1}(x)$. THE POINT (a, b) IS THE REFLECTION OF THE POINT (b, a) ACROSS THE LINE $y = x$, SO THE GRAPH $y = f^{-1}(x)$ IS THE **MIRROR IMAGE** OF THE GRAPH $y = f(x)$ REFLECTED ACROSS THE LINE $y = x$.



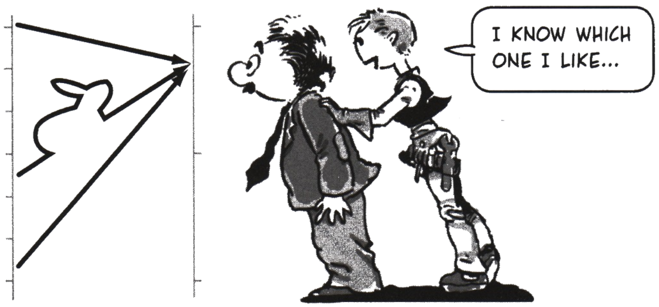






HERE ARE TWO EXAMPLES: ABOVE, THE GRAPH $y = x^3$ AND ITS INVERSE THE CUBE ROOT, AND ON THE RIGHT THE SUPER-IMPORTANT NATURAL LOGARITHM AND ITS INVERSE THE EXPONENTIAL.

CAN WE INVERT A FUNCTION THAT IS NOT ONE-TO-ONE, THAT GOES UP AND DOWN? IF MANY ARROWS LAND AT A POINT y, WHICH ONE DO WE REVERSE? THE ANSWER IS: PICK WHICHEVER ONE YOU LIKE AND IGNORE THE REST!



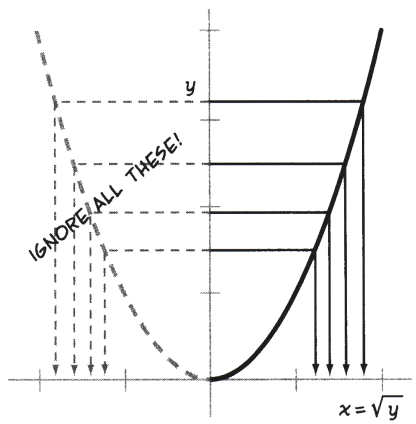
ONE SYSTEMATIC WAY TO DO THIS IS TO FLIP ONLY ARROWS ORIGINATING ON AN **INTERVAL WHERE THE FUNCTION IS ONE-TO-ONE.** FOR EXAMPLE, $f(x) = x^2$ IS INCREASING (AND SO ONE-TO-ONE) ON THE INTERVAL $[0, \infty)$. REVERSING ONLY THE ARROWS THAT START THERE MAKES AN INVERSE

$$f^{-1}(x) = \sqrt{x}$$

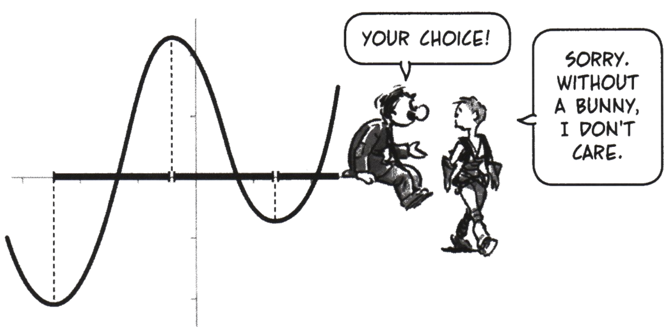
THAT ALWAYS GIVES THE **NON-NEGATIVE SQUARE ROOT.** THEN FOR ALL $x \geq 0$,

$$f(f^{-1}(x)) = x$$

$$f^{-1}(f(x)) = x \quad \text{(NO NEGATIVE } x \text{ ALLOWED!)}$$



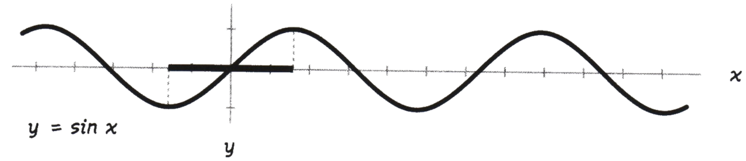
THIS WORKS FOR ANY FUNCTION f: **RESTRICT ITS DOMAIN** TO AN INTERVAL WHERE f IS INCREASING (OR DECREASING), AND ON THIS INTERVAL, f HAS AN INVERSE.



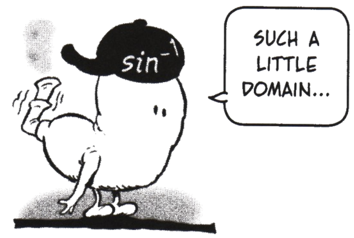
Excerpts from Gonick, Larry (2012). *The Cartoon Guide to Calculus*. New York, NY. Harper Collins.

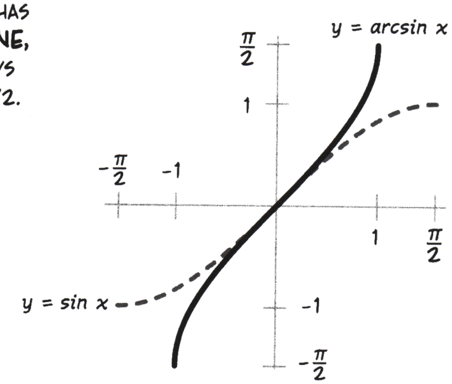
Second Big, Important Example:
Inverse Circular Functions

THE SINE AND COSINE WOBBLE UP AND DOWN, UP AND DOWN... BUT ON SOME SHORT INTERVALS, THEY ARE INCREASING! LET'S CONCENTRATE ON THE SINE, BECAUSE THE COSINE WORKS EXACTLY THE SAME WAY. YOU CAN SEE THAT THE SINE INCREASES ON THE INTERVAL $[-\frac{\pi}{2}, \frac{\pi}{2}]$ WHERE ITS VALUES RISE FROM -1 TO 1.



RESTRICTED TO THAT INTERVAL, THE SINE HAS AN INVERSE FUNCTION, CALLED THE **ARCSINE,** WITH DOMAIN $[-1, 1]$. THE ARCSINE ALWAYS TAKES ON VALUES BETWEEN $-\pi/2$ AND $\pi/2$.





WHY IS IT CALLED THE **ARC**SINE? BECAUSE IT'S THE ARC LENGTH CORRESPONDING TO A GIVEN SINE.

$$\text{IF } \sin\theta = y \quad \text{THEN } \theta = \arcsin y$$

θ IS AN ANGLE WHOSE SINE IS y. THIS ANGLE, BEING MEASURED IN RADIANS, IS THE LENGTH OF THE CORRESPONDING ARC ON THE UNIT CIRCLE (SEE P. 35). OTHER ANGLES HAVE THE SAME SINE, BUT θ IS THE **ONLY** ANGLE BETWEEN $-\pi/2$ AND $\pi/2$ WITH $\sin\theta = y$.